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AN ALGORITHM FOR COMPUTING
THE ALPHA-WIDTH OF $(0,1)$ MATRICES

WALTER L. STANLEY

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ALPHA-WIDTH OF $(0,1)$ MATRICES

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Walter L. Stanley

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ALPHA-WIDTH OF $(0,1)$ MATRICES

by

Walter L. Stanley

Lieutenant, United States Navy

Submitted in partial fulfillment of
the requirements for the degree of

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ABSTRACT

A branch and bound technique is used to derive an algorithm for computing the alpha-width of any matrix of zeros and ones. Through computation of the 1-width of over 200 matrices of various dimensions, it is found that less than 20 minutes of computation time on the Control Data 1604 digital computer is required to complete the computation for most matrices. Applications of the algorithm to integer programming and to various targeting problems are described. Extensions are suggested for computing the minimal cost alpha-width, and for computing a minimal C-cover.

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TABLE OF SYMBOLS

$x \in X$	x is a member of X
$\sum_{j=1}^n x_j$	$x_1 + x_2 + \dots + x_n$
$(a_1, a_2, \dots, a_n)^T$	The <u>transpose</u> of the matrix (or vector), $(a_1, a_2, \dots, a_n)$ $(a_1, \dots, a_n)^T = \begin{matrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{matrix}$
$A \prec B$	A is <u>majorized</u> by B
$A \not\prec B$	A is <u>not majorized</u> by B
$\overline{A}$	The <u>maximal</u> matrix with row sums R
$[e_{ij}]$	The matrix, E
$\overrightarrow{0}$	The vector, $(0, 0, \dots, 0)$
$B \sim A$	The <u>relative complement</u> of A with respect to B
$A \cup B$	The <u>union</u> of A and B
$A \cap B$	The <u>intersection</u> of A and B
$X, \overline{X}$	A <u>partition</u> of a set Y satisfying: $X \cap \overline{X} = \emptyset$ $X \cup \overline{X} = Y$
$\binom{n}{k}$	The binomial coefficient. $\binom{n}{k} = n!/(k!(n-k)!)$
$p_1 \& p_2$	p_1 <u>and</u> p_2 (logical)
$p_1 + p_2$	p_1 <u>or</u> p_2 (logical) section 7.3 only
$\overline{p_1}$	<u>not</u> p_1 (logical)

1. A Targeting Problem.

Consider the following rather specialized targeting problem: a communications network is given (Fig. 1), in which stations can communicate directly only with those stations to which they are connected by a link. Of course, this would be the case with any kind of land line network, but it is possible also, in the case of UHF radio communications, micro-wave relay systems, and even signal light. We ask this question: what is the minimum number of stations that must be destroyed so that the network is totally disrupted; that is, so that no pair of surviving stations can communicate?

The answer is given in Figure 2; in which those stations targeted have been crossed out. It is perhaps surprising to note that the station most central to the network; the one directly connected to the greatest number of stations, is not targeted. In fact, if this station were included in the target list, we should be forced to target the four indicated targets anyway, and thus we would have been forced away from the optimal solution.

Let us note, parenthetically, that no claim is made that our targeting policy is the best one. It is quite probably valid, and indeed optimal, if the purpose of the attack is, for example, the total (and temporary) disruption of an enemy's warning system for the protection of a second strike to follow immediately. But assume that the network is a railroad system. It is quite possible that a policy of bombing junctions, switching yards, and accessible rail lines would have little lasting effect on the effectiveness of the transportation system because of the ready availability of repair

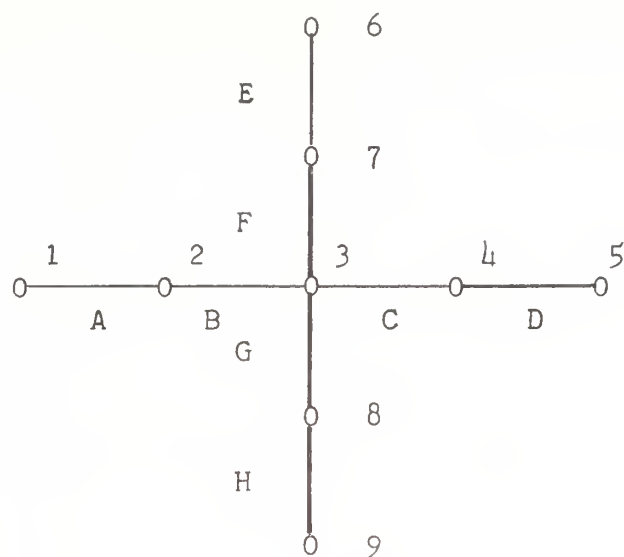


Figure 1

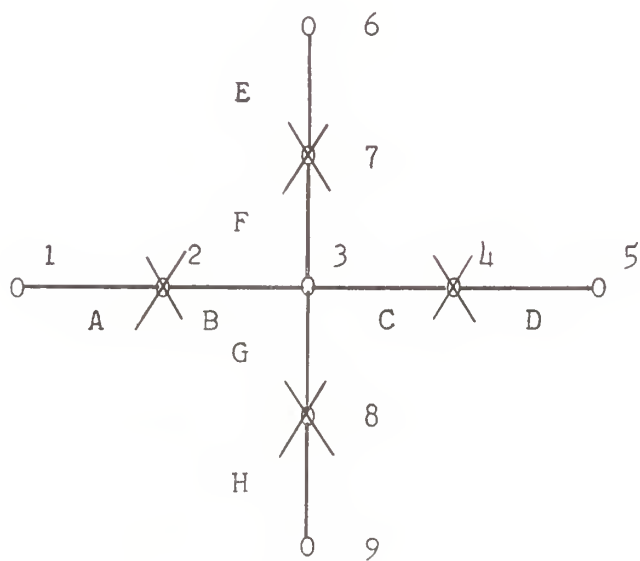


Figure 2

equipment and personnel. For example, in the interdiction of the French railroads prior to the invasion of Normandy in World War II; our bombing of marshalling yards and other junctions caused little disruption of rail traffic, although it did strain the repair capabilities of the rail system severely. On the other hand, when bridges over the Seine, Oise, and Meuse Rivers were added to the target list, results were spectacular. On 26 May, all routes over the Seine north of Paris were closed to rail traffic and remained closed for the next thirty days. By contrast, marshalling yards could be repaired in one or two days. (See pp 217-230; and especially p 228 of [6]).

No matter, this simple problem will serve to illustrate the very general algorithm to be described in section 3; without requiring that cumbersome set-up procedures be learned before getting down to work.

The solution to this targeting problem was obtained without difficulty after the initial error of trying to include the central station (number 3) in the target list, merely by inspection of the network layout. It is unfortunate that so few communications networks of nine stations and eight connecting links are of interest in a problem of this type. Clearly, if a network of interesting size were examined (let us say on the order of 15 stations and 35 connecting links), the solution by inspection would be quite difficult. Where, then, are we to look for a method of attack on this problem?

It is well known from the theory of graphs, that every graph may be represented by an incidence matrix of zeros and ones; in fact

by any of several incidence matrices depending upon the purpose for the representation. [1]. For the purpose of this paper we will use the following terminology from graph theory: a node of a graph is the junction of two or more links of the graph (synonym: vertex); an arc is a link between two nodes, and in this paper will be considered to be without direction. We define the node-arc incidence matrix, A, of a graph, by construction as follows: List the nodes of the graph horizontally and the arcs vertically so that they are labels of columns and rows of the matrix, respectively. If the j^{th} node is a terminal point of the i^{th} arc, set $a_{ij} = 1$. Otherwise, set $a_{ij} = 0$. The node-arc incidence matrix of the communications network of Figure 1 is displayed in Figure 3.

The targeting problem restated in graph theoretic terms is: Find the minimum number of nodes so that each arc of the graph has at least one of the nodes as a terminal. Since we already know the answer to this simple problem, it would be well to examine this solution applied to the node-arc incidence matrix. We construct a new matrix from the incidence matrix by including only those columns labelled with one of the nodes in the solution set. This matrix is displayed in Figure 4. It contains the same number of rows as the original matrix, but has only four columns. We note that whereas there were two "1"'s in each of the rows of the incidence matrix (one for each terminal of each arc); there is only one "1" in the sub-matrix.

A little reflection upon the above observation leads to a third formulation of the targeting problem: given the node-arc incidence matrix of the communications network, find the smallest subset of

	1	2	3	4	5	6	7	8	9
A	1	1	0	0	0	0	0	0	0
B	0	1	1	0	0	0	0	0	0
C	0	0	1	1	0	0	0	0	0
D	0	0	0	1	1	0	0	0	0
E	0	0	0	0	0	1	1	0	0
F	0	0	1	0	0	0	1	0	0
G	0	0	1	0	0	0	0	1	0
H	0	0	0	0	0	0	0	1	1

Figure 3

	2	4	7	8
A	1	0	0	0
B	1	0	0	0
C	0	1	0	0
D	0	1	0	0
E	0	0	1	0
F	0	0	1	0
G	0	0	0	1
H	0	0	0	1

Figure 4

columns of the matrix with the property that each row is represented by at least one "1" in this subset of columns. But this smallest subset of columns is precisely what Fulkerson and Ryser call a minimal set of representatives for the $(0,1)$ matrix, A ; and the cardinality of this set is called the "width" of A . [4]. The problem may be generalized: we require that each row of the matrix be represented by at least α "1"'s (where α is a positive integer). We shall use the terminology "minimal α -set of representatives for the $(0,1)$ matrix, A "; and " α -width of A ". This terminology is due also to Fulkerson and Ryser. [4].

Thus the simple targeting problem may be solved by finding the 1-width of the node-arc incidence matrix of the communications network. It is the purpose of this paper to present an algorithm for finding the α -width of any $(0,1)$ matrix; and for specifying at least one minimal α -set of representatives for that matrix. Since we already have solved one problem of this type, we shall use this communications network and its associated incidence matrix for illustrative purposes throughout the balance of this paper.

We now state the general problem which we desire to solve: given a finite set, X , and a class, Y , of k non-empty subsets of X (but not necessarily the class of all non-empty subsets of X), find a subclass, Z , of Y , with the property that if $x \in X$, then x is a member of at least α of the members of Z . This is a quite general problem, as will be shown in later sections of this paper. Any problem which can be formulated in the terms specified in this paragraph is capable of being solved by the algorithm to be presented. The incidence matrix for this abstract problem is constructed by listing members of

X vertically and subsets of X horizontally. Then we place a "1" in the i^{th} row and j^{th} column if the i^{th} member of X is a member of the j^{th} subset of X.

2. The Class, $\mathcal{U}(R,S)$.

Let A denote the (0,1) matrix of size m by n; that is, A is a matrix with m rows and n columns, each of whose elements is either zero or one. Let the sum of all of the elements of the i^{th} row be denoted by r_i ; and the sum of all of the elements in the j^{th} column be denoted by s_j . That is:

$$(1) \quad \sum_{j=1}^n a_{ij} = r_i \quad (i=1, \dots, m)$$

$$(2) \quad \sum_{i=1}^m a_{ij} = s_j \quad (j=1, 2, \dots, n)$$

We note that $\sum_{i=1}^m r_i = \sum_{j=1}^n s_j$.

We call the column vector, $(r_1, r_2, \dots, r_m)^T = R$, the row sum vector; and the row vector, $(s_1, s_2, \dots, s_n) = S$, the column sum vector. We denote by $\mathcal{U}(R,S)$ the class of all (0,1) matrices of size m by n with row sum vector, and column sum vector, R and S, respectively.

From the class, $\mathcal{U}(R,S)$, many very interesting combinatorial results may be obtained. An excellent survey of this material may be found in Ryser.[10]. We will be concerned primarily with a parameter, $\tilde{\epsilon}$, or $\tilde{\epsilon}(\alpha)$, of the class, which is defined as the greatest lower bound on the α -width of any matrix in $\mathcal{U}(R,S)$. That is, $\tilde{\epsilon}$ is the α -width of the matrix in $\mathcal{U}(R,S)$ which has the smallest α -width of any matrix in the class.

Although not of concern until a later section, it will be of interest to determine under what conditions the class, $\mathcal{A}(R, S)$ is non-empty. Let $\delta_i = (1, 1, \dots, 1, 1, 0, 0, \dots, 0)$ be an n -dimensional vector with the first r_i components equal to one, and the remaining $n - r_i$ components equal to zero. We then define a matrix of the form,

$$\bar{A} = \begin{bmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_m \end{bmatrix}$$

called the maximal matrix with row sum vector, R . It has column sum vector, $\bar{S} = (\bar{s}_1, \bar{s}_2, \dots, \bar{s}_n)$. Now since $\sum_{i=1}^m r_i = \sum_{j=1}^n \bar{s}_j$; for R fixed, $\bar{S}$ is unique, by definition of the δ_i , and the class $\mathcal{A}(R, S)$, by a simple contradiction argument, has only one member; namely, $\bar{A}$.

Let $Q = (q_1, q_2, \dots, q_k)$ and $Q^* = (q_1^*, q_2^*, \dots, q_k^*)$ be any two k -dimensional vectors whose components are non-negative integers. We say that Q is majorized by Q^* , denoted $Q \prec Q^*$, provided that with subscripts renumbered so that $q_1 \geq q_2 \geq \dots \geq q_k$; and $q_1^* \geq q_2^* \geq \dots \geq q_k^*$, the following statements are true:

$$(3) \quad q_1 + q_2 + \dots + q_j \leq q_1^* + q_2^* + \dots + q_j^* \quad (j=1, 2, \dots, k-1)$$

$$(4) \quad q_1 + q_2 + \dots + q_k = q_1^* + q_2^* + \dots + q_k^*$$

We say that Q is normalized if $q_1 \geq q_2 \geq \dots \geq q_k$. These two definitions now enable us to give conditions under which $\mathcal{A}(R, S)$ is non-empty.

Theorem 2.1

Let $R = (r_1, r_2, \dots, r_m)$, and $S = (s_1, s_2, \dots, s_n)$ be two normalized vectors whose components are non-negative integers, and such that $\sum_{i=1}^m r_i = \sum_{j=1}^n s_j$. Let $\bar{A}$ be the maximal matrix of size m by n , with row sum vector, R , and column sum vector $\bar{S} = (\bar{s}_1, \bar{s}_2, \dots, \bar{s}_n)$.

Then a necessary and sufficient condition that $\mathcal{A}(R, S)$ be non-empty is that $S \prec \bar{S}$.

Proof: Assume $S \not\prec \bar{S}$. Since $\sum_{j=1}^n s_j = \sum_{j=1}^n \bar{s}_j = \sum_{i=1}^m r_i$, it must be that $S \not\prec S$ because equation (3), above, is violated, that is, for some k , it must be the case that $s_1 + s_2 + \dots + s_k > \bar{s}_1 + \bar{s}_2 + \dots + \bar{s}_k$. But then $\bar{A}$ is not maximal, since the first k columns of A contain more "1"'s than the first k columns of $\bar{A}$. The hypothesis is that A is maximal, so we have arrived at a contradiction, thus demonstrating the necessity of the theorem.

To show sufficiency; we shall construct a matrix, $A \in \mathcal{A}(R, S)$ from the matrix $\bar{A}$. This construction is due to Ryser. [9]. The construction will proceed by shifting ones in the i^{th} row of $\bar{A}$ to other positions in the same row. We note again, that R, S , and $\bar{S}$ are all normalized, and that $S \prec \bar{S}$. If $s_1 < \bar{s}_1$, rearrange the ones in the rows of A so that only s_1 ones remain in the first column. We may do this unless $s_j > s_1$ ($j=2, \dots, n$), in which case, $\bar{s}_1 + \bar{s}_2 + \dots + \bar{s}_n > n \cdot s_1 \geq s_1 + s_2 + \dots + s_n = \bar{s}_1 + \dots + \bar{s}_n$; an absurdity. We continue by induction. Suppose that the first t columns of A have been rearranged. The matrix thus far constructed has the form,

$$A' = \begin{bmatrix} b_1 & b_2 & \dots & b_t & b_{t+1} & \dots & b_n \end{bmatrix}$$

where there are s_j ones in the j^{th} column of A' ($j = 1, \dots, t$).

We now construct the $(t+1)^{\text{st}}$ column. Let the number of ones in the j^{th} column be $s_j^!$ ($j = t+1, \dots, n$). We may construct A' without loss of generality such that, $s_{t+1}^! \geq \dots \geq s_n^!$. Now it is possible that either $s_{t+1} < s_{t+1}^!$, or that $s_{t+1} > s_{t+1}^!$. We consider each case in turn;

Case I: $s_{t+1} < s_{t+1}^!$

Remove ones from the $(t+1)^{\text{st}}$ column, placing them in other columns to the right. If sufficiently many ones may be removed by this procedure, the column of A is constructed, and we are finished. Suppose therefore, that there remain, d ones in column $t+1$, so that $s_{t+1} < d \leq s_{t+1}^!$. Let the matrix at this stage be denoted by $[e_{rs}]$. Now if $d > s_{t+1}$, then for every $e_{r,t+1} = 1$; we must have $e_{rj} = 1$ ($j = t+2, \dots, n$). Hence $s_{t+1} + \dots + s_n$ must at least equal $d \cdot (n-t)$. But $s_{t+1} < d$; $s_{t+2} \leq s_{t+1} < d$; etc., so that

$$s_{t+1} + \dots + s_n < (n-t) \cdot d \leq s_{t+1} + \dots + s_n$$

an absurdity.

Case II: $s_{t+1} > s_{t+1}^!$.

Insert ones in the $(t+1)^{\text{st}}$ column from columns to the right. If sufficiently many ones can be inserted, we are finished. We therefore assume that sufficiently many ones cannot be inserted by this procedure; in fact, we assume that column $t+1$ contains only d ones such that $s_{t+1}^! \leq d < s_{t+1}$. Again, let the matrix at this stage of construction be denoted by $[e_{rs}]$. Then if $e_{r,t+1} = 0$, it must be the case that $e_{rj} = 0$ ($j = t+1, \dots, n$). Now suppose that $e_{qj} = 1$ for some $j \geq t+2$. Then either $e_{qk} = 1$ for all $k \leq t+1$, or else, for some $k \leq t$, $e_{qk} = 0$. Consider the case in which $e_{qk} = 0$. Since $s_k \geq s_{t+1} > d$, there must exist $e_{pk} = 1$, and $e_{q,t+1} = 0$. We interchange e_{qj} and e_{qk} ; and also interchange e_{pk} and $e_{p,t+1}$. This increases the value of d by one without changing the value of any column sum for columns to the left of column $t+1$. Suppose we make all such interchanges and still, $d < s_{t+1}$. This situation includes the case mentioned above, that $e_{qk} = 1$ for all $k \leq t+1$. It is no longer possible to shift ones from columns $t+2, \dots, n$; into columns $1, \dots, t+1$. This must mean that

either all of the ones for a given row are in columns to the left of column $t+2$; or that all of elements of a given row to the left of column $t+2$ are equal to one. In either case, it must be that,

$$s_1 + \dots + s_t + d = s_1 + \dots + s_t + s_{t+1}$$

But then, since $S \prec \bar{S}$,

$$s_1 + \dots + s_{t+1} \leq \bar{s}_1 + \dots + \bar{s}_{t+1} \quad (= s_1 + \dots + s_t + d)$$

whence $s_{t+1} \leq d$; contrary to the assumption. QED

We now consider an extension of the concept of α -width. Let $C = (c_1, \dots, c_m)$ be an m -dimensional vector of non-negative integers. We wish to find the smallest subset of columns of $A \in \mathcal{H}(R, S)$ such that the i^{th} row of A is represented by at least c_i ones in this subset of columns. Such a subset of columns will be called a minimal C -cover for A , and we shall denote its cardinality by $\epsilon(C)$, called the C -width of A . Clearly, if $C = (\alpha, \dots, \alpha)$, then $\epsilon(C) = \epsilon(\alpha)$. We define $\tilde{\epsilon}(C)$ to be the greatest lower bound on the C -width of any matrix in $\mathcal{H}(R, S)$. $\tilde{\epsilon}(C)$ can be estimated by $\rho(C)$ as follows:

$$(5) \quad \rho(C) = \text{the smallest integer such that } \sum_{j=1}^{\rho} s_j \geq \sum_{i=1}^m c_i.$$

We shall use this formula in the algorithm to be presented in section three. Note that if $C = (\alpha, \dots, \alpha)$; then $\rho(C) =$ the smallest integer such that $\sum_{j=1}^{\rho} s_j \geq m \cdot \alpha$

3. Derivation of the Algorithm.

We shall now describe our algorithm for finding the α -width of a $(0,1)$ matrix. The branch and bound technique was suggested to me by D. R. Fulkerson of the RAND Corporation, Santa Monica, California, and is patterned after the branch and bound solution to the travelling salesman problem designed by Little, et alii. [7]

We have several techniques for estimating $\epsilon(\alpha)$ (which we shall henceforth call the α -width of the class, $\mathcal{W}_\alpha(R,S)$). One such technique is described in the preceding section, in which we compute the parameter, ρ . Now a given $(0,1)$ matrix of size m by n , is a member of a class, $\mathcal{W}_\alpha(R,S)$. We can partition the class, $\mathcal{W}_\alpha(R,S)$ into two sub-classes, one consisting of those matrices which have a selected column, say column p , as a member of a minimal α -set; and the other consisting of those matrices for which column p is not a member of any minimal α -set for the matrix.

We thus have two sub-classes, each of which has no more members than the original class, and we know that the original matrix must be in one, and only one of the sub-classes. Consider the sub-class whose matrices have column p as a member of a minimal α -set. We may use this information to reduce the dimensions of all the matrices in the class as follows: if $\alpha = 1$, then every row which has a one in column p is adequately represented by column p , and needs not be considered subsequently. If $\alpha \neq 1$, we still may note that these same rows are represented once by column p , and thus need be represented only $\alpha - 1$ more times subsequently. Furthermore, we have made a "decision" about column p , namely that it is included in a minimal α -set of all matrices in this sub-class. We may thus reduce the dimensions of all matrices of the sub-class by one column, and (if $\alpha \neq 1$) by a number of rows. If $\alpha \neq 1$, we will keep track of those rows which yet need only $\alpha - 1$ representatives. Hence we have for this class a vector, C , whose components are either α , or $\alpha - 1$.

We may also reduce the dimensions of the matrices in the other sub-class by one column, for we have made a "decision" for this sub-class

namely, that no α -set contains column p , for any matrix of the sub-class. Hence every row of the matrices of this sub-class needs to be subsequently represented α times, regardless of the value of α .

Now let us estimate $\tilde{\epsilon}$ for each of the two sub-classes. It is clear that these two numbers are both estimates of the α -width of the original matrix. We may actually improve the estimate for the first sub-class discussed, by adding one to the estimate of $\tilde{\epsilon}$ for the sub-class. This is to account for the inclusion of column p in any α -set for any matrix in this sub-class. Now, it is certain that the smaller of these two numbers is not greater than the α -width of the original matrix.

Let us examine the sub-class corresponding to the smaller of the two estimates. We may partition this sub-class into two sub-classes, and so forth, until finally, some sub-class will be so small as to contain a unique matrix whose C -width we can determine by inspection. Part of such a continuing procedure is represented by the tree structure of Figure 5.

Now at any point in the procedure, the set of junctions (Fig. 5) which have no lines leading toward another junction represent a partition of the class to which the original matrix belongs into two or more sub-classes. By an obvious extension of the above discussion, the smallest of the several estimates for the α -width of the original matrix is not larger than the α -width of the original matrix. We may then focus our attention on the sub-class corresponding to this smallest estimate, branching out from the corresponding junction until one of the earlier estimates of ϵ is smaller than any of the most recently constructed estimates. Now let us set up a formal algorithm based upon the preceding discussion.

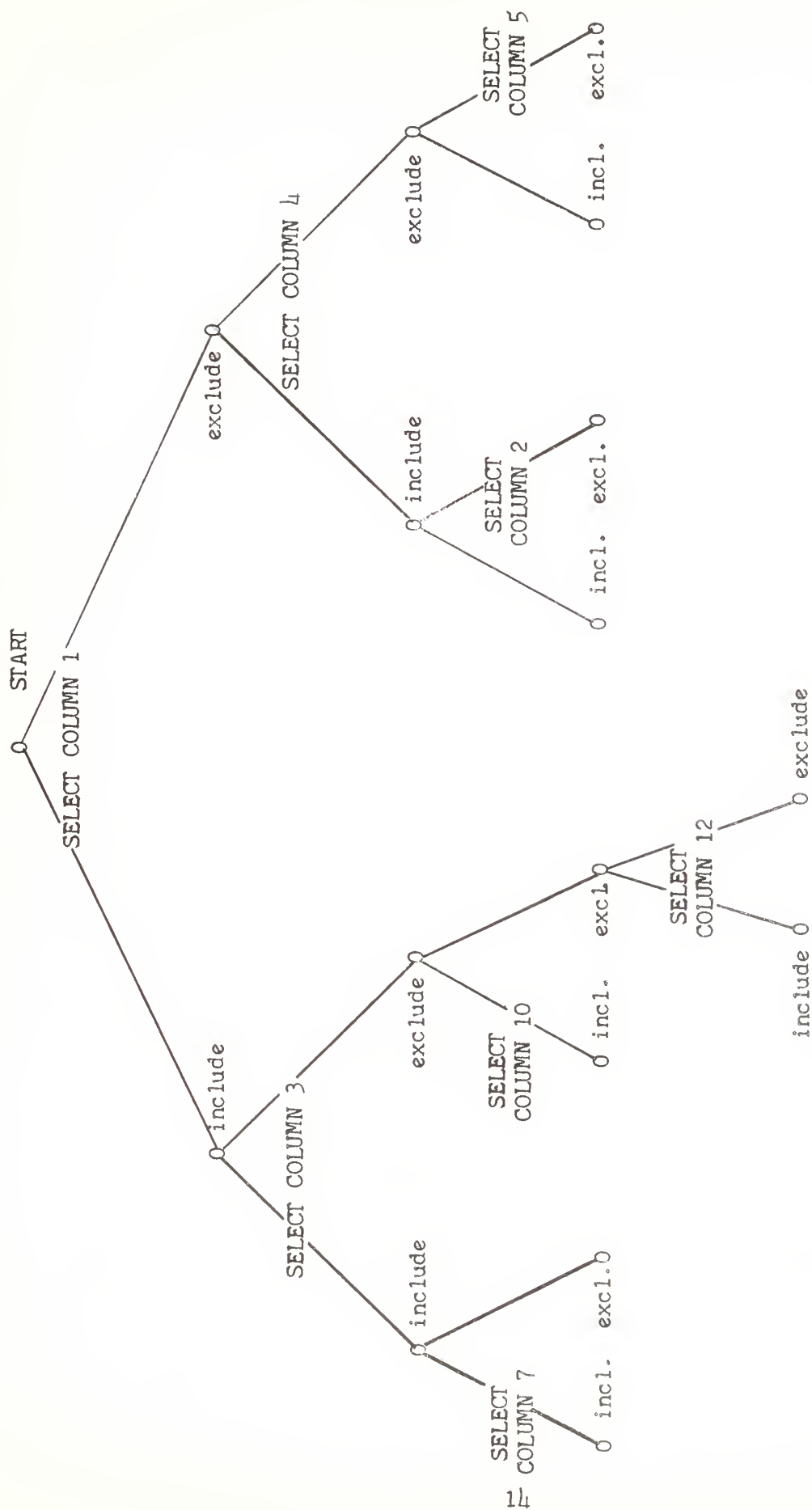


Figure 5

Let the $(0,1)$ matrix, $A \in \mathcal{A}(R,S)$, be given, with dimensions m by n . A matrix is said to be normalized when both its row sum and column sum vectors are normalized, and when the elements of the matrix have been rearranged so as to fit the new row and column sum vectors. Clearly, we lose no generality by considering only normalized matrices. Therefore, throughout the remainder of this paper, we assume that all matrices and sub-matrices have been normalized as part of the operation of constructing them.

Notation will, of necessity, become rather cumbersome, and for that reason, we now present such notation as we shall need in this section. There will be certain preliminary steps which serve to decrease the amount of work required in the main part of the algorithm, and since these preliminary steps are not always applicable, we shall assume that the given matrix, $A \in \mathcal{A}(R,S)$, is the one with which we shall enter the main part of the algorithm.

The procedure in the algorithm is basically broken into two parts; (1) selecting a column for inspection and deriving the two sub-classes corresponding to the inclusion in, and the exclusion from the α -set of the selected column (the "branch" portion); and (2) estimating ϵ from each sub-class and choosing among all estimates, the smallest for the next iteration (the "bound" portion). We shall carry out the "bound" portion of the procedure by calculating ρ for each of the sub-classes and adding to ρ , the number of columns previously included in the α -set on the current branch. Eventually we shall obtain a sub-class of matrices, one of whose dimensions is zero, and is thus, empty. Clearly, $\tilde{\epsilon}$ for this sub-class is zero. We can make a test for completion at this point. If the test fails, we

continue the algorithm along some other branch. It can be seen that we shall derive an even number of different sub-matrices of A before we reach termination. We shall subscript these sub-matrices in the order in which they are derived. Associated with each of the sub-matrices, of course, will be a row sum vector, a column sum vector, an estimate of the C-width of the class to which the sub-matrix belongs, and an estimate of the α -width of A based upon the condition that it can be obtained by continuing along the branch from which we derived the present sub-matrix. Note that since we may discontinue consideration of one branch at any time, and return to a previously discontinued branch; the subscripts of the matrices which we shall derive at any point of the procedure bear no relation to the subscript of the matrix from which the derivation follows. This point will be made again during our step by step description of the algorithm.

Now, we subscript every parameter associated with a particular sub-matrix with the same sub-script as its associated sub-matrix. We shall also require a "label" for each sub-matrix, and the typical label will be of the form " $a, \overline{b}, \overline{c}, \overline{d}, \dots$ ". This label gives us the information that for each particular sub-matrix, every column of A which is present in the label has been branched upon; and those columns which appear unbarred are assumed to be included in the α -set, whereas those which appear with a bar over them are assumed to be excluded from the α -set. Thus, the a, b, c, d in the example label above represent positive integers which are the column numbers of the original A matrix. One further convention to which we shall adhere; an even sub-script is taken to mean that the latest column upon which we branched is considered to be included in the α -set associated with the sub-matrix,

and thus this column number will appear unbarred in the associated label. On the other hand, an odd subscript is taken to mean that the latest column branched upon is considered to be excluded from the α -set associated with the sub-matrix, and thus this column number will appear in the associated label with a bar over it. The described notation is summarized below:

$A_p \in \mathcal{M}(R_p, S_p)$ has dimensions m_p by n_p .

$$R_p = (r_{p1}, r_{p2}, \dots, r_{pm_p})^T.$$

$$S_p = (s_{p1}, s_{p2}, \dots, s_{pn_p}).$$

$$\rho_p = \text{the minimum } k \text{ such that } \sum_{j=1}^k s_{pj} \geq \sum_{i=1}^m c_{pi}$$

$$\epsilon_p' = \rho_p + \text{the number of columns unbarred in the label of } A_p.$$

It is obvious that much information must be recorded for each of several matrices. Although a structure similar to that of Figure 5 could be used, we suggest the format of Figure 6. This figure shows a typical matrix A_p and all of the required information associated with this matrix. It will be convenient to suppress zero elements of the matrix. Note that we list the subscripts of the columns of the original matrix along the top, and directly below that, the order of subscripts for the derived matrix, A_p . The order of subscripts for the rows of A_p is listed along the left side of the matrix, and R_p and S_p are listed along the right side and the bottom, respectively. At some convenient point we list ρ_p , ϵ_p' , and the label associated with the matrix.

In section four, we solve the targeting problem of section one using this algorithm. The reader may desire to read section four

concurrently with the description of the algorithm which follows.

3.1 Preliminary Steps.

P1. If $r_m < \alpha$; the α -width does not exist. We terminate, or else decide to look for an α -width in which α is a smaller integer than that which the original problem specified.

P2. If $r_m > \alpha$; go directly to step S1, in the main part of the algorithm.

	3	10	1	2	5	(column subscripts of A)	
	1	2	3	4	5	R_p	
1	1		1	1	1	4	
2	1	1	1			3	
3	1	1		1		3	
4		1			1	2	$\rho_p = 2$
5			1			1	
6				1		1	$\epsilon_p = 5$
S_p	3	3	3	3	2		"4,7,6,8,9"

Figure 6

P3. If $r_m = r_{m-1} = \dots = r_{m-k} = \alpha$, for some k , ($0 \leq k \leq m-1$); then it is evident that each "1" in any of these $k+1$ rows must belong to a column in the minimal α -set of representatives for A. Therefore, in each such row, say the i^{th} , for each j such that $a_{ij} = 1$, record that the j^{th} column is in the minimal α -set and delete the j^{th} column from the matrix. Let $C = (\alpha, \dots, \alpha)$ be an m -dimensional vector. For each k such that $a_{kj} = 1$ subtract one from the k^{th} component of C .

When this has been done for all columns, j , deleted from the matrix, delete any row, i , for which $c_i \leq 0$. Finally, recompute new row sum and column sum vectors, normalize the new matrix, and proceed to step S1 in the main portion of the algorithm. Now the set of columns that has been deleted in this preliminary step will not again be explicitly mentioned. The reader is cautioned to remember to add these columns to the α -set computed in the next section in order to arrive at the true α -width of the matrix, A .

3.2 The Branch and Bound Algorithm.

S1. We are given $A \in \mathcal{U}(R, S)$ which has been normalized. If C was not computed in step P3, let $C = (\alpha, \dots, \alpha)$. Cross out column one of A . We shall branch on this column because it is the column with the largest column sum. This is an entirely arbitrary decision. We could branch on any column whatsoever, but it seems reasonable that the one with the largest column sum would be likely to be included in the α -set. A counterexample is easy to construct. In any case, it is now necessary to decide whether or not to include this column in the α -set.

S2. Let us denote the matrix, A by $[\delta_1 \ \delta_2 \ \dots \ \delta_n]$. Construct $A_1 (= [\delta_2 \ \delta_3 \ \dots \ \delta_n])$ and label it "1". We examine the consequences of excluding column one from the α -set. A_1 is of size m_1 by $n_1 (=n-1)$, and $A_1 \in \mathcal{U}(R_1, S_1)$. Our decision means that we still must locate c_i representatives for each row, but that we may not use any of the "1"'s in the excluded column of A . Calculate ρ_1 by equation (2-5), and since no columns are unbarred in the label, let $\epsilon_1^! = \rho_1$.

S3. We next examine the consequences of including column one in the α -set. Construct A_2 by deleting column one from A , forming a

temporary R_2 vector, and deleting every row for which the component-wise difference of R and R_2 is greater than or equal to the corresponding component of C . That is, if $r_i - r_{2i} \geq c_i$; delete row i . Clearly this can happen at this step only if $c_i = 1$. We have now reduced A by one column and perhaps some number of rows. This reduced matrix, when normalized is called A_2 , and we now form the permanent vectors, R_2 and S_2 . We label this sub-matrix, "1". ρ_2 is calculated by equation (2-5), and since column one is unbarred in the associated label, $\varepsilon_2' = \rho_2 + 1$.

S4. We must now decide along which branch it will be most profitable to continue. We make the decision by choosing the sub-matrix associated with $\min [\varepsilon_1', \varepsilon_2']$. If $\varepsilon_1' = \varepsilon_2'$, the choice is arbitrary. When using the algorithm for hand computation, the best choice is probably that matrix with the greatest number of unbarred columns in the associated label, that is, in this case, A_2 . Having made this decision, we set $\min [\varepsilon_1', \varepsilon_2'] = \infty$, so that the same branch will not be chosen again at a later stage. We proceed to step S5. All succeeding steps in the algorithm will be described in general terms.

S5. In the preceeding step, we decided to proceed using matrix A_L , say, with associated label, " $p, q, \bar{r}, s, \bar{t}, \bar{u}$ "; a particular one of the k matrices thus far constructed ($k \geq L$). Since A_L has been normalized, the first column has the largest column sum. We therefore select this column as the next branch point. Let us say that this column corresponds to the v^{th} column of A .

S6. Denoting A_L by $[\delta_{L1}, \delta_{L2}, \dots, \delta_{Lm_L}]$, we construct the next sub-matrix, A_{k+1} ($= [\delta_{L2}, \delta_{L3}, \dots, \delta_{Lm_L}]$), thus finding the sub-matrix

corresponding to a decision to exclude column one of A_L (column v of A) from the α -set. A_{k+1} is of size m_{k+1} by n_{k+1} , and $A_{k+1} \in \mathcal{A}(R_{k+1}, S_{k+1})$.

S7. Noting that columns p , q , and s of A have been included up to this stage, we form a vector of row sums of included columns, which we shall call RS . That is, referring back to the A matrix, we compute for each row, the number of ones in columns p , q , and s . Obviously, for this particular label, the sum cannot exceed three. Now we compute a test vector, RT . Let the i^{th} component of RT be the maximum of zero and $c_i - rs_i$ (the i^{th} component of RS). The vector, RT , gives us the number of ones yet to be included in each row of the matrix, by some subsequent choice of columns. Since both RS and RT are vectors which are required only at this branch, and will thence be discarded, there is no need to subscript them.

S8. Returning to our decision to exclude column one of A_L , we examine each component of R_{k+1} . If any component of R_{k+1} is less than its corresponding component of RT , it is infeasible to exclude this column. We set $\epsilon_{k+1}^! = \infty$, but retain the label of A_{k+1} which is, in this case, " $\underline{p}, \underline{q}, \underline{r}, \underline{s}, \underline{t}, \underline{u}, \underline{v}$ ". We then proceed to the next step. If, on the other hand, we determine that the label represents a feasible set of columns, that is, no component of R_{k+1} is less than its corresponding component of RT , we compute ρ_{k+1} by equation (2-5) using the vector RT in place of C . Since there are three unbarred columns in this typical label, we set $\epsilon_{k+1}^! = \rho_{k+1} + 3$.

S9. We now construct matrix A_{k+2} with label " $\underline{p}, \underline{q}, \underline{r}, \underline{s}, \underline{t}, \underline{u}, \underline{v}$ ". This matrix corresponds to the decision to include column v of A in the α -set. We delete column one of A_1 . For every row which had a

"1" column one of A_L , we subtract one from the appropriate component of RT . If this component now is zero, we delete the corresponding row of A_L . When the procedure is completed, we have the matrix A_{k+2} . We compute R_{k+2} and S_{k+2} ; and finally, $\epsilon_{k+2}^!$ which equals $\rho_{k+2} + 1$ in this case. If, however, either dimension of the matrix becomes zero at this step, we proceed to step S11, as it is possible that this represents termination. ρ_{k+2} , of course, is computed using the vector RT instead of C .

S10. Let P be the $k+2$ dimensional vector whose components are the $\epsilon_i^!$. We find the minimum component of P , choosing arbitrarily in case of a tie, and use this component's corresponding matrix for our next path. Although an arbitrary choice in case of a tie will lead to solution, there are two techniques for choosing between branches that will probably shorten the algorithm somewhat. These are, either to stay with the current branch in case of a tie in which the current branch is involved; or to take the branch which has the largest number of unbarred columns in its label. The second method is probably the best, but in the computer algorithm we shall use neither technique; branching instead on the matrix with the smallest sub-script because of programming simplicity. Let us say we have chosen matrix A_L for our branching matrix. We set $\epsilon_L^! = \infty$, and return to step S5, continuing the algorithm.

S11. Since A_{k+2} is of zero dimension; $\rho_{k+2} = 0$. Then $\epsilon_{k+2}^!$ is equal to the number of columns that are unbarred in the label of A_{k+2} . Now if $\epsilon_{k+2}^! > \epsilon_i^!$ for any $i < k+2$, we have not necessarily found a solution, so we return to step S10, after duly recording the proper values for all of the parameters associated with this sub-matrix.

Of course, it is now meaningless to consider row sum and column sum vectors. This is a minor point, since the only purpose of these vectors is in computing ρ , and in deriving subsequent matrices. It is clear, though, that if we at a later time choose to branch upon this matrix of zero dimension, it is because we have found that it is after all, an optimal solution to our problem.

If, on the other hand, $\epsilon_{k+2}^i \leq \epsilon_i^1$ for all $i \leq k+2$, the unbarred columns in the label of A_{k+2} constitute a minimal α -set of representatives for A , and the cardinality of this set of columns is the α -width of A . Thus we have arrived at a termination point of the algorithm. In the next sub-section, we shall prove that the algorithm does find a minimal α -set of representatives, and that it terminates in a finite number of steps.

3.3 Proof that a Solution is Reached.

We need to show that the algorithm does find a minimal α -set of representatives even though many possible combinations of columns have not been considered. It is first necessary, though, to show conditions under which the α -width exists. We have already stated, in step P1, that if $\alpha < r_m$, the α -width does not exist. We now prove a necessary and sufficient condition for the existence of the more general C-cover of A :

Theorem 3.1.

The matrix, A , has a C-width for every vector, C , whose components, c_i are bounded above by r_i .

Proof: By hypothesis, $c_i \leq r_i$; hence A , itself, is a C-cover for every admissible vector, C . For a fixed C , the collection of all

C-covers is thus non-empty, and clearly is finite. Then the collection has a minimal member, and the cardinality of this minimal member is the C-width of A.

QED

Corollary:

The matrix, A, has an α -width, $\epsilon(\alpha)$, for each integer α in the interval, $1 \leq \alpha \leq r_m$.

We shall now demonstrate that the branch and bound technique of section 3.2 will find the minimal C-cover of A in a finite number of iterations. We shall further show, that the branch and bound technique is independent of the technique for computing a bound on ϵ , under some rather simple restrictions. We shall call $\tilde{\epsilon}(C)$, the C-width of the class, $\mathcal{A}(R, S)$. Now $\tilde{\epsilon}(C)$ is clearly a function of the dimensions of the matrices in the class. Let $\rho^*(C)$ be an estimate of $\tilde{\epsilon}(C)$ such that $\rho^*(C) \leq \tilde{\epsilon}(C)$, and such that for the class $\mathcal{A}$, one of whose dimensions is zero, $\rho^*(\vec{0}) = \epsilon(\vec{0}) = 0$; where $\vec{0}$ is the m dimensional zero vector. We insist in what follows that the estimating technique for computing $\rho^*(C)$ be applied consistently. The parameter, ρ , described in section two satisfies the above requirements on $\rho^*(C)$.

Let A be a given matrix and estimate $\tilde{\epsilon}(C)$ by $\rho^*(C)$. Then the C-width of A is not less than $\rho^*(C)$. Now construct matrices A_1 and A_2 as in steps S6 and S9 of section 3.2 using any column of A, say column t, instead of that column whose sum is the largest. Estimate $\tilde{\epsilon}(C_i)$ for each of the sub-matrices thus constructed by $\rho_1^*(C_1)$, and $\rho_2^*(C_2)$ respectively. Then the C_1 -width of A_1 is not less than $\rho_1^*(C_1)$,

and the C_2 -width of A_2 is not less than $\rho_2^*(C_2)$. The vectors, C_i , have components equal to the number of "1"'s yet necessary to represent the i^{th} row of A . For example, if the i^{th} component of C were 2 and the selected column contained a one in its i^{th} place, then the i^{th} component of C_1 would be 2; but the i^{th} component of C_2 would be 1. Now since column t of A must be either included in, or excluded from the minimal C -cover of A , the C -width of A is not less than $\min [\rho_1^*(C_1), \rho_2^*(C_2)+1]$. We need no longer consider $\rho^*(C)$ as an estimate of the C -width of A . Clearly then, if we repeat this estimating process, using A_1 or A_2 as a new given matrix according to whether $\rho_1^*(C_1)$ or $\rho_2^*(C_2)+1$ is the smaller, we may compute two additional estimates of the C -width of A . Eventually (after a finite number of such estimates have been made), we shall construct a matrix, one of whose dimensions is zero. In that case, $\rho_{2k}^*(C_{2k}) = 0$, and the C -width of A cannot be less than the cardinality of the set of columns slated for inclusion in the C -cover of A . This set of columns is, in fact, a C -cover, and if the cardinality of this set is less than or equal to all of the other computed estimates of the C -width of A , then it is a minimal C -cover, since we required that any estimate be bounded above by $\tilde{\epsilon}(C)$.

We refer the reader once again to the scheme illustrated in Figure 5. If each branch of this tree were to be taken to its termination (at worst, the point at which each column of A had been tested either for inclusion or exclusion), each such terminal could be represented by an n -tuple as follows: let the i^{th} component be one if the i^{th} column had been included on this branch, and let it be zero otherwise. There are 2^n unique n -tuples, hence at most 2^n

corresponding terminals, each attainable in a finite number of steps. Hence the algorithm must terminate in a finite number of steps.

4. Manual Computation with the Algorithm.

Let us return to the targeting problem described in section one, and solve this problem to illustrate the use of the branch and bound algorithm. We reproduce the matrix of Figure 3, as Figure 7 for ready reference. Zeros have been suppressed, and we have appended the components of R and S to the right and bottom of the matrix, respectively. Figure 8 depicts the normalized matrix, A. We have appended the original column subscripts above the matrix.

In this example, $\alpha = 1$; and it should be noted that in general, increasing α , significantly increases the complexity of the manual algorithm, because of short-cuts used in deciding which rows may be deleted. These short-cuts are not available for $\alpha > 1$. The RT vector need not be constructed, since its components could only be zero or one, and such a simple vector can be handled by inspection. However, the short-cuts cannot be conveniently programmed, so the computer version of the algorithm can handle differing α 's with almost equal facility.

	1	2	3	4	5	6	7	8	9	R
A	1	1								2
B		1	1							2
C			1	1						2
D				1	1					2
E						1	1			2
F			1				1			2
G			1					1		2
H								1	1	2
S	1	2	4	2	1	1	2	2	1	

Figure 7

	3	2	4	7	8	1	5	6	9	
	1	2	3	4	5	6	7	8	9	R
A		1								2
B	1	1								2
C	1		1							2
D			1				1			2
E				1				1		2
F	1			1						2
G	1				1					2
H					1				1	2
S	4	2	2	2	2	1	1	1	1	

Figure 8

We show, in Figures 9 and 10, the matrices A_1 and A_2 , respectively, derived from A as follows: We delete column 1 from A and since this column corresponds to column 3 of the original matrix, we label A_1 , "3". Naturally, we have normalized both R_1 and S_1 . Now locate each row of A which has a "1" in the first column. Delete this row, delete column 1, and we now have A_2 , after normalizing R_2 and S_2 . This criterion for deleting rows is a simplification of computing RT , which is the short-cut mentioned at the beginning of this section. The rows deleted in the example are rows B, C, G, and F. We label A_2 , "3".

For the matrix, A_1 , $\rho_1 = 4$, since $\sum_{i=1}^4 s_{1i} = m_1 = 8$. Similarly, for A_2 , $\rho_2 = 4$, since $\sum_{i=1}^4 s_{2i} = m_2 = 4$. Hence $\epsilon'_1 = \rho_1 = 4$; and $\epsilon'_2 = \rho'_2 + 1 = 5$.

Min $[\epsilon_1^1, \epsilon_2^1] = \epsilon_1^1$ so we choose to branch on matrix A_1 . We set $\epsilon_1^1 = \infty$. Column 1 of A_1 corresponds to column 2 of the incidence matrix, and is the column which has the largest column sum.

	2	4	7	8	1	5	6	9	
	1	2	3	4	5	6	7	8	R_1
A	1				1				2
D		1				1			2
E			1				1		2
H				1				1	2
B	1								1
C		1							1
F			1						1
G				1					1
S_1	2	2	2	2	1	1	1	1	
Sub-matrix, A_1 . "3" $\rho_1 = 4$. $\epsilon_1^1 = 4$									

Figure 9

	2	4	7	8	1	5	6	9	
	1	2	3	4	5	6	7	8	R_2
A	1				1				2
D		1				1			2
E			1				1		2
H				1				1	2
S_2	1	1	1	1	1	1	1	1	
Sub-matrix, A_2 . "3" $\rho_2 = 4$. $\epsilon_2^1 = 5$									

Figure 10

We delete column one of A_1 and thus have matrix A_3 , which we label " $\overline{3}, 2$ ". This sub-matrix is shown in Figure 11, below. Since row B of A_1 has row sum zero in A_3 , and since there are no included columns in the label, this represents an infeasible set of column exclusions. Therefore, without further consideration, we set $\epsilon_3^1 = \infty$.

	4	7	8	1	5	6	9	
	1	2	3	4	5	6	7	R_3
D	1				1			2
E		1				1		2
H			1				1	2
A				1				1
C	1							1
F		1						1
G			1					1
S_3	2	2	2	1	1	1	1	
Sub-matrix A_3 . " $\overline{3}, 2$ " $\epsilon_3^1 = \infty$								

Figure 11

Next, we delete each row of A_1 which has a "1" in the first column, namely, rows A and B; and we delete the first column of A_1 . This gives us matrix A_4 , depicted on the next page in Figure 12. Of course, the label for A_4 is " $\overline{3}, 2$ ". Now $\sum_{i=1}^3 s_{4i} = m_4 = 6$; so $\rho_4 = 3$, and $\epsilon_4^1 = \rho_4 + 1 = 4$. Note that the sum of column 4 of A_1 goes to zero in A_4 , so we may delete it.

Now $\min [\epsilon_1^1, \epsilon_2^1, \epsilon_3^1, \epsilon_4^1] = \epsilon_4^1 = 4$. Hence we choose to branch next on matrix A_4 . We set $\epsilon_4^1 = \infty$, and choose column one of A_4 for examination.

	4	7	8	5	6	9	
	1	2	3	4	5	6	R_4
D	1			1			2
E		1			1		2
H			1			1	2
C	1						1
F		1					1
G			1				1
S_4	2	2	2	1	1	1	
Sub-matrix A_4 . "3,2" $\rho_4 = 3$ $\epsilon_4^1 = 4$							

Figure 12

Deleting this column, which corresponds to column 4 of the original matrix, produces A_5 , with label "3,2,4". This sub-matrix is reproduced in Figure 13, below. Note that the sum of row C of A_4 has gone to zero in A_5 .

	7	8	5	6	9	
	1	2	3	4	5	R_5
E	1			1		2
H		1			1	2
F	1					1
G		1				1
D			1			1
S_5	2	2	1	1	1	
Sub-matrix A_5 . "3,2,4" $\epsilon_5^1 = \infty$						

Figure 13

This means that the label represents an infeasible combination of columns, and we therefore set $\epsilon_5^! = \infty$ without further consideration of this sub-matrix.

Now we derive sub-matrix A_6 by deleting rows D and C from A_4 , since each of these rows has a "1" in column one of A_4 . We also delete column one of A_4 and give this sub-matrix the label, "3,2,4". The matrix is presented in Figure 14 below. Since $\sum_{i=1}^2 s_{6i} = m_6 = 4$, we have that $\rho_6 = 2$, and $\epsilon_6^! = 4$, since there are two unbarred columns in the label for A_6 . Clearly $\epsilon_6^! = \min [\epsilon_i^!] (i=1, \dots, 6)$; so we choose to continue along this branch. Thus A_6 will be our next branching matrix.

	7	8	6	9	
	1	2	3	4	R_6
E	1		1		2
H		1		1	2
F	1				1
G		1			1
S_6	2	2	1	1	
Sub-matrix A_6 .	"3,2,4"		$\rho_6 = 2$	$\epsilon_6^! = 4$	

Figure 14

We derive sub-matrix A_7 from A_6 (see Figure 15) by deleting column one of A_6 , corresponding to column 7 of the incidence matrix. This sub-matrix has label, "3,2,4,7". Once again we run into the situation of a row sum going to zero, so this sub-matrix, too, represents an infeasible combination of columns. Therefore, we set $\epsilon_7^! = \infty$, and proceed.

	8	6	9	
	1	2	3	R_7
H	1		1	2
G	1			1
E		1		1
S_7	2	1	1	
Sub-matrix A_7 .	<u>"3,2,4,7"</u>			$\epsilon_7^! = \infty$

Figure 15

By the deletion of rows E and F from A_6 , we arrive at sub-matrix A_8 , which also has column one of A_6 deleted. This two by two matrix is displayed in Figure 16, below. The label is "3,2,4,7", and we note that $s_{8,1} = m_8 = 2$; therefore, $\rho_8 = 1$, and since there are three unbarred columns in the label, $\epsilon_8^! = 4$. Note also that the column sum of column 6 of the original matrix has gone to zero, so we delete that column also. We see that $\epsilon_8^! = \min [\epsilon_i^!]$ ($i=1, \dots, 8$), so we branch on matrix A_8 . We remember to set $\epsilon_8^! = \infty$, and choose column one of A_8 for examination. This column corresponds to column 8 of the original matrix.

	8	9	
	1	2	R_8
H	1	1	2
G	1		1
S_8	2	1	
Sub-matrix A_8	<u>"3,2,4,7"</u>		$\rho_8 = 1 \quad \epsilon_8^! = 4$

Figure 16

Deletion of column one of A_8 gives us a one by one sub-matrix which has label " $\overline{3,2,4,7,8}$ ". This matrix represents an infeasible combination of columns since row G has vanished. Thus we set $\epsilon_9^1 = \infty$, and proceed.

We see immediately that we have reached termination, since the matrix A_{10} is of zero dimension and has label " $\overline{3,2,4,7,8}$ ". This means that $\epsilon_{10}^1 = 4$ and that $\epsilon_{10}^i = \min [\epsilon_1^i]$ ($i=1, \dots, 10$).

The 1-width of the incidence matrix is 4, and a minimal 1-set of representatives for the incidence matrix is the set of columns, (2,4,7,8). These, of course, would be the station numbers that were to be targeted in our original problem.

5. Computation of $\tilde{\epsilon}(\alpha)$.

We notice that for the very simple problem presented in section four, ten matrices had to be written down. The writer has observed that in hand computation, one matrix can be used for deriving only two or three sub-matrices before the paper becomes impossible to read. Even a small matrix requires a considerable amount of time to write down, especially when normalization cannot be done in one's head. In the computer version, due to limited storage space it is necessary to recompute a matrix each time it must be used, so even at high digital computer speeds, it would be desirable to reduce as far as possible the number of matrices that had to be examined.

Unfortunately, for the computation by hand, little can be done to simplify the problem, but in the case of the computer algorithm, it is possible to compute $\tilde{\epsilon}(\alpha)$ exactly at little expense in time. Unfortunately, this computation will be useful only when $\alpha = 1$; and

hence, when the components of the RT vector of section three can be only zero or one. But this case is the one which is of greatest interest, and it is thus very worthwhile to study this computation.

There are at least two derivations possible. One, which is entirely combinatorial in nature, gives considerable insight into the class, $\mathcal{U}(R,S)$, at the expense of being quite lengthy and not very intuitive. The interested reader is referred to Fulkerson and Ryser. [4].

We shall use a network derivation which is considerably shorter and more intuitive. The procedure for $\alpha = 1$ is outlined in [2]. It should be noted that the formula was first derived using network considerations. We require the following theorem in the network derivation to follow:

Theorem 5.1.

Let $A \in \mathcal{U}(R,S)$ have α -width, $\epsilon(\alpha)$. Then there is at least one matrix, A_ϵ , in $\mathcal{U}(R,S)$ such that the first ϵ columns of A constitute a minimal α -set of representatives for A_ϵ .

Proof: Consider any matrix, $A \in \mathcal{U}(R,S)$ with α -width, $\epsilon(\alpha)$. Let E^* be that subset of the columns of A consisting only of the members of the minimal α -set of representatives. If E^* is the first $\epsilon(\alpha)$ columns of A , then $A = A_\epsilon$. Therefore we assume that column p is the leftmost column of A not in E^* . Now locate column k such that column k is the rightmost column of A in E^* .

Let $R_E = (r_{E1}, \dots, r_{Em})$ be the vector of row sums of E^* . Now if $r_{Ei} = \alpha$ and there is no $a_{ip} = 0$ for which $a_{ik} = 1$, ($i=1, \dots, m$), we may replace column k by column p in E^* , and the new columns of E^* are a minimal α -set of representatives. Suppose therefore, that we have for some i , $r_{Ei} = \alpha$, $a_{ip} = 0$ and $a_{ik} = 1$. We call a_{ik} a critical

one of E^* . Then since $s_p \geq s_k$, there must be an $a_{jp} = 1$ for which $a_{jk} = 0$. ($j \neq 1$). Further, for each critical one in column k , there is a distinct one in column p with a corresponding zero in column k . We perform interchanges on such critical ones, the typical interchange resulting in $a_{ip} = 1$; $a_{ik} = 0$; $a_{jp} = 0$; and $a_{jk} = 1$. We may now replace column k by column p in E^* and the new columns of E^* form a minimal α -set of representatives.

Clearly this construction is possible for each column to the left of column ϵ , which is not in E^* . Hence the construction yields

$$A_\epsilon \in \mathcal{A}(R, S). \quad \text{QED}$$

5.1 A Supply-Demand Network.

In this section we shall consider directed networks. Let N represent the set of nodes of a network and $\mathcal{Q}$ represent the set of arcs. We denote an arc between x and y , members of N ; by the ordered pair, (x, y) and assert that the notation implies the arc is directed from x to y , and is not the same arc as the one denoted (y, x) . We associate with each arc in $\mathcal{Q}$, a non-negative function $c(x, y)$ called a capacity function, and a non-negative function $f(x, y)$ called a flow function. We associate with some nodes in N a non-negative function $a(x)$ which may be thought of as a supply of some commodity available at node x , and we associate with some other nodes in N , a non-negative function $b(y)$ which may be thought of as a demand for some commodity by node y .

We make use of the following shorthand notation, Let S, T , be subsets of N , and let x, y be elements of N . Then by $c(S, x)$ we mean $\sum_{s \in S} c(s, x)$, and similarly for $f(S, x)$. Also, by $c(S, T)$ we mean $\sum_{s \in S} \sum_{t \in T} c(s, t)$, and similarly for $f(S, T)$. Analogous shorthand will be used for the functions, a and b .

Now let us assume we have a class of matrices, $\mathcal{M}(R,S)$. We devise a network for this class as follows: Let there be n nodes denoted $b_1, \dots, b_n$; with demand function, $b(b_j) = s_j$, the j^{th} component of S . Let there be m nodes denoted $a_1, \dots, a_m$; with supply function $a(a_i) = r_i$, the i^{th} component of R . Let $B = b_j$; $A = a_i$. Let $(a_i, b_j) \in \mathcal{Q}$ for all i, j . Let $c(a_i, b_j) = 1$ for all i, j . This network has an arc capacity of one for each of the $m \cdot n$ elements of a matrix, $A \in \mathcal{M}(R,S)$. The commodity available at the nodes of A , and required by the nodes of B is, of course, "1"'s to distribute among these $m \cdot n$ elements of the associated class of matrices. We construct a flow in the network satisfying the following constraints:

$$(6) \quad f(x, N) - f(N, x) \leq a(x) \quad x \in A$$

$$(7) \quad f(N, x) - f(x, N) \geq b(x) \quad x \in B$$

$$(8) \quad 0 \leq f(x, y) \leq c(x, y) \quad (x, y) \in \mathcal{Q}$$

Clearly this construction is possible. For $R = (2, 2, 2, 2)$ and $S = (3, 3, 2)$ such a network has been constructed in Figure 17. The number by each arc is the value of the flow function for that arc. Now let us construct the corresponding matrix (Figure 18). If $f(a_i, b_j) = 1$, let $a_{ij} = 1$; if $f(a_i, b_j) = 0$, let $a_{ij} = 0$. This matrix, A , is in the class, $\mathcal{M}(R,S)$. Furthermore, each unique feasible flow corresponds to a unique matrix, $A \in \mathcal{M}(R,S)$, and conversely.

Then let us ask this question: under what conditions can we construct a flow so that $f(a_i, T) \geq \alpha$ for each i , and for $T = \{b_1, \dots, b_\epsilon\}$? This flow would correspond to distributing at least α ones from each a_i to the nodes corresponding to the first ϵ columns of a matrix in $\mathcal{M}(R,S)$. If we can locate the smallest ϵ for which this flow is feasible, we shall have found $\tilde{\epsilon}$. See Theorem 5.1.

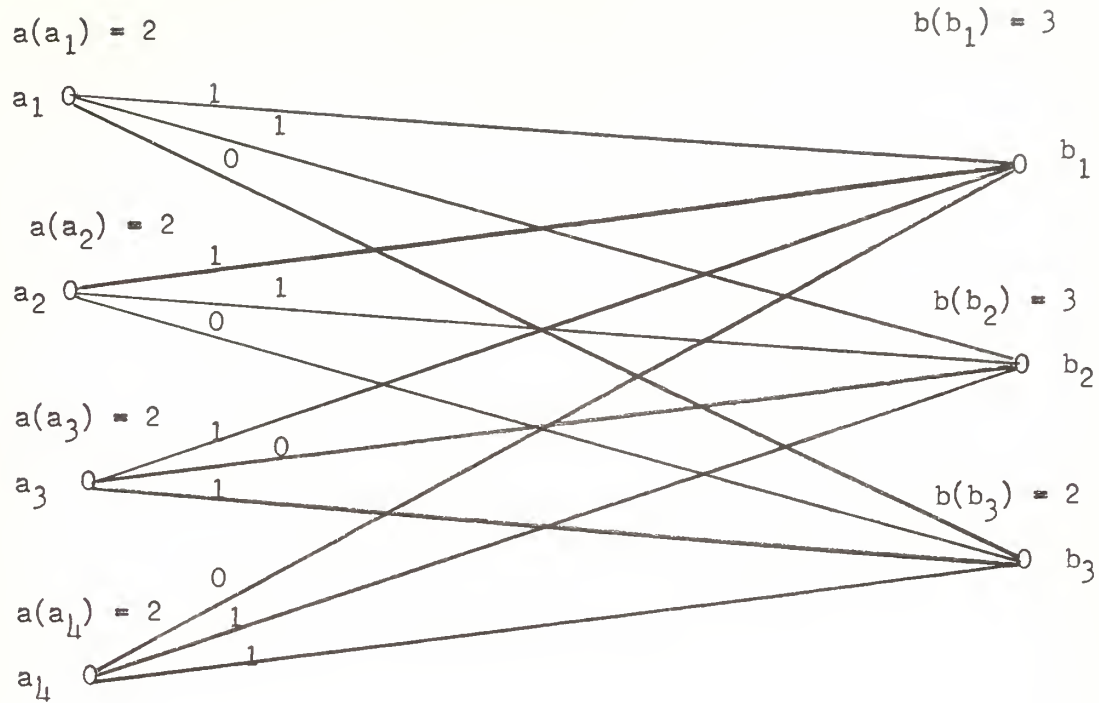


Figure 17

1	1	0	2
1	1	0	2
1	0	1	2
0	1	1	2
3	3	2	

Figure 18

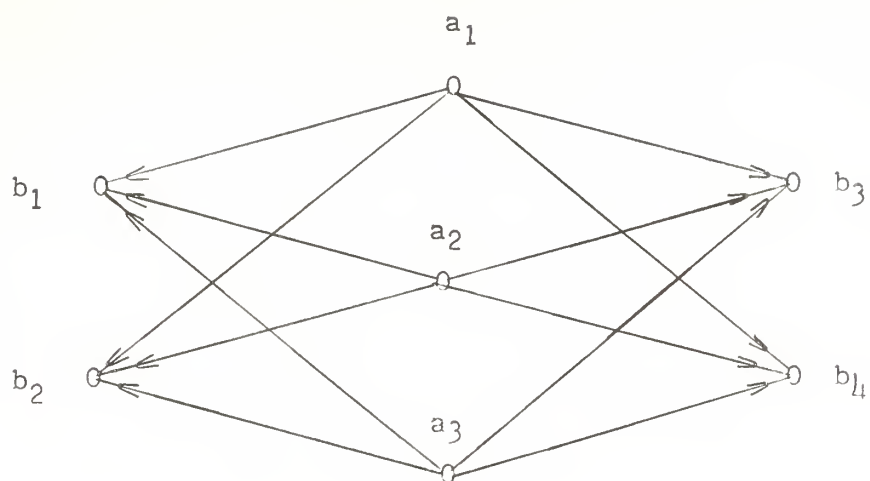


Figure 19

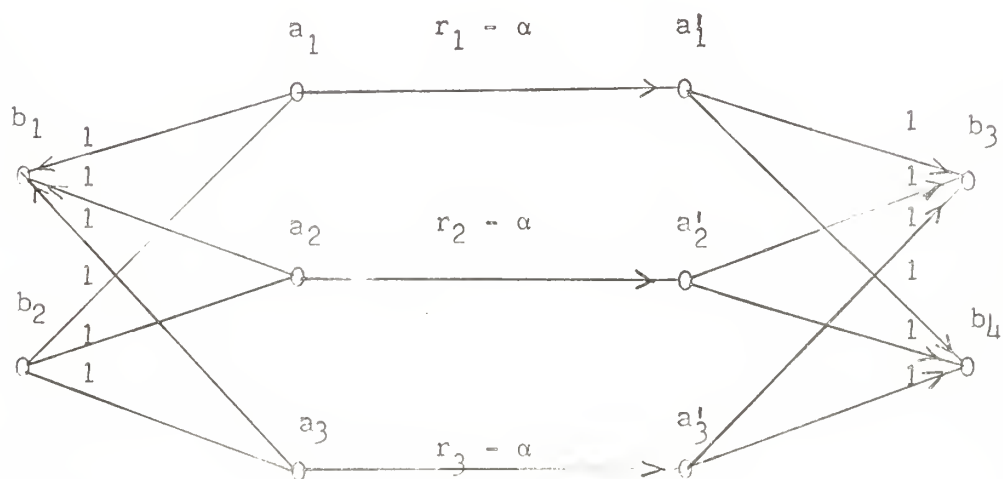


Figure 20

Since $\sum_{i=1}^m r_i = \sum_{j=1}^n s_j$; for the demands to be satisfied at each b_j , the supply at the nodes of A must be totally exhausted for a feasible flow. Consider the network of Figure 19, with four sink nodes (those with positive demand functions), and three source nodes (those with positive supply functions). Let us construct for each $\epsilon \leq n$; a network as follows: Let B_ϵ be the first ϵ nodes of B . Let $(x,y) \in \mathcal{Q}$ for $x \in A$ and $y \in B_\epsilon$. Now construct m new nodes $(a_1^!, \dots, a_m^!)$ and let the set of these nodes be called $A^!$. Let $(a_i, a_i^!) \in \mathcal{Q}$ for each $i \leq m$. Let $(x,y) \in \mathcal{Q}$ for $x \in A^!$ and $y \in B \sim B_\epsilon$ (the relative complement of B_ϵ with respect to B). Let the following be true:

$$\begin{array}{ll}
 a(a_i) = r_i & i = 1, \dots, m \\
 b(b_j) = s_j & j = 1, \dots, n \\
 c(a_i, b_j) = c(a_i^!, b_k) & \begin{array}{l} i = i, \dots, m \\ j = 1, \dots, \epsilon \\ k = \epsilon+1, \dots, n \end{array} \\
 c(a_i, a_i^!) = r_i - \alpha & i = 1, \dots, m
 \end{array}$$

The construction corresponding to the network of Figure 19 for $\epsilon = 2$ is shown in Figure 20 on the preceding page. Numbers above each arc are the capacities of the arc. What we have done is this: since the capacity of each $(a_i, a_i^!)$ is α units less than the supply at a_i , at least α units of supply must be distributed to the nodes of B_ϵ . We call a flow feasible if and only if constraints (6), (7), and (8) are satisfied and if

$$(9) \quad f(x, N) - f(N, x) = 0 \quad x \in A^!$$

is also satisfied. Clearly, the smallest ϵ for which a feasible flow exists in this type of network is $\tilde{\epsilon}(\alpha)$.

Theorem 5.2.

The constraints

- $$\begin{aligned}
(6) \quad & f(x, N) - f(N, x) \leq a(x) & x \in A \\
(7) \quad & f(N, x) - f(x, N) \geq b(x) & x \in B \\
(8) \quad & 0 \leq f(x, y) \leq c(x, y) & (x, y) \in \mathcal{Q} \\
(9) \quad & f(x, N) - f(N, x) = 0 & x \in A'
\end{aligned}$$

where $a(x) \geq 0$, $b(x) \geq 0$; are feasible if and only if,

$$(10) \quad b(B \cap \bar{X}) - a(A \cap \bar{X}) \leq c(X, \bar{X})$$

holds for every partition of N into subsets X and $\bar{X}$ ($= N \sim X$).

This is the well known supply-demand theorem due to Gale. A proof may be found in [2].

We apply (10) to our network (in general) and observe that for partitions of the form:

$$\begin{aligned}
X &= \{a_1, \dots, a_m; a_1', \dots, a_e'; b_1, \dots, b_e; b_{f+1}, \dots, b_n\} \text{ and} \\
\bar{X} &= \{a_{e+1}', \dots, a_m'; b_{e+1}, \dots, b_f\}
\end{aligned}$$

where e and f are integer parameters satisfying

$$0 \leq e \leq m; \quad e \leq f \leq n$$

(10) is of the form:

$$(11) \quad s_{e+1} + \dots + s_f \leq (r_{e+1} - \alpha) + \dots + (r_m - \alpha) + e \cdot (f - e)$$

The validity of the inequality is obvious except possibly for the term, $e \cdot (f - e)$. This term is merely the number of a_i' in X , times the number of b_j in X ; and is the total capacity of all arcs connecting these two sets of nodes.

Theorem 5.3.

The constraints, (6), (7), (8), and (9) of Theorem 5.2 are feasible for a network of the type of Figure 20, and for fixed e , if and only if (11) holds for all permissible values of e and f .

The proof of this theorem consists of looking at all subsets of nodes not of the form on the preceding page, and verifying that Theorem 5.2 is valid for these subsets if it is valid for the subsets of the above form. Since the proof is not very interesting, and rather lengthy, it is omitted.

Theorem 5.3 assures us that we need not test all subsets of nodes with (10) in order to assure ourselves that we have a feasible flow. Now let us multiply (11) by minus one; and rearrange terms: We arrive at:

$$(12) \quad r_{e+1} + \dots + r_m - (s_{\epsilon+1} + \dots + s_f) + e \cdot (f - \epsilon) \geq \alpha \cdot (m - e)$$

We thus have the condition that $\tilde{\epsilon}$ is the smallest integer for which (12) is satisfied for all integer values of e and f in the ranges:

$$0 \leq e \leq m \qquad \epsilon \leq f \leq n$$

Now the left side of (12) is the class invariant which Fulkerson and Ryser call $N(\epsilon, e, f)$. [4]. For ease of computation we define a function, $Q(\epsilon, e, f) = N(\epsilon, e, f) - \alpha \cdot (m - e)$. $\tilde{\epsilon}$ is the smallest ϵ for which

$$(13) \quad Q(\epsilon, e, f) \geq 0 \qquad 0 \leq e \leq m; \qquad \epsilon \leq f \leq n$$

We take first differences with respect to ϵ , e and f ; and derive the following recursion formulas:

$$(14) \quad Q(\epsilon+1, e, f) = Q(\epsilon, e, f) + s_{\epsilon+1} - e$$

$$(15) \quad Q(\epsilon, e+1, f) = Q(\epsilon, e, f) + f + \alpha - \epsilon - r_{e+1}$$

$$(16) \quad Q(\epsilon, e, f+1) = Q(\epsilon, e, f) + e - s_{f+1}$$

Since we know that ρ is a lower bound on the α -width of any $A \in \mathcal{U}(R, S)$ we may take $\epsilon = \rho$, and compute an $m+1$ by $n-\epsilon$ array making liberal use of (15) and (16). If one of the numbers is negative, we increment ϵ by one using (14), and compute the $m+1$ by $n-\epsilon$ array

for this new value of ϵ . When an array is found which contains no negative members, we have found $\tilde{\epsilon}$.

Now, as we mentioned before, we do not advise this procedure for hand computation, and it cannot be used for α greater than one, but the case $\alpha = 1$ is the most important case by far, as will be seen in section seven, and a digital computer is admirably suited to perform these simple arithmetic computations. In the next section we shall discuss the computer program in which the above formula was used; and the results obtained with a large number of matrices.

6. The Algorithm Program.

We present a procedural flow chart for the branch and bound algorithm in Figure 21. As much as possible of the procedure is described in abbreviated, but intuitive language. Where variable names are necessary they are either the names given to the same variables used in Sections 3.1 and 3.2, or they are defined on the flow chart itself near the point at which they are first used. Variable names are used that are in reasonable agreement with the corresponding names used in the computer program.

The algorithm was programmed for the Control Data 1604 computer using FORTRAN 63 source language, and CODAP 1 assembly language. The assembled program is included in this paper as Appendix I. There are several features of the program which deserve discussion.

Since we are dealing with matrices composed of zeros and ones, storage space can be conserved by letting a single bit represent an element of the matrix. Then we may use logical operations to manipulate the matrix. Normally this would require writing the entire program at

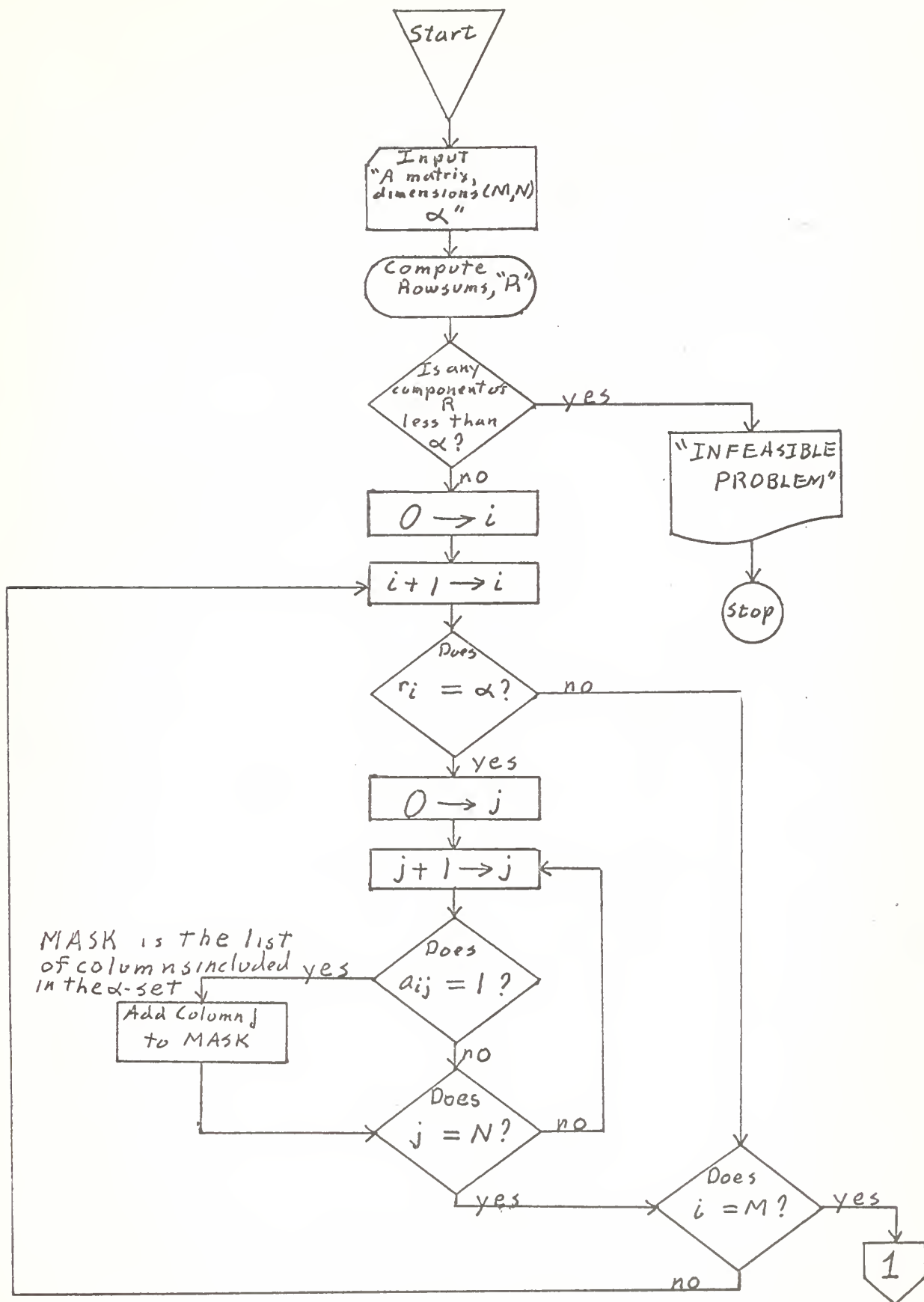
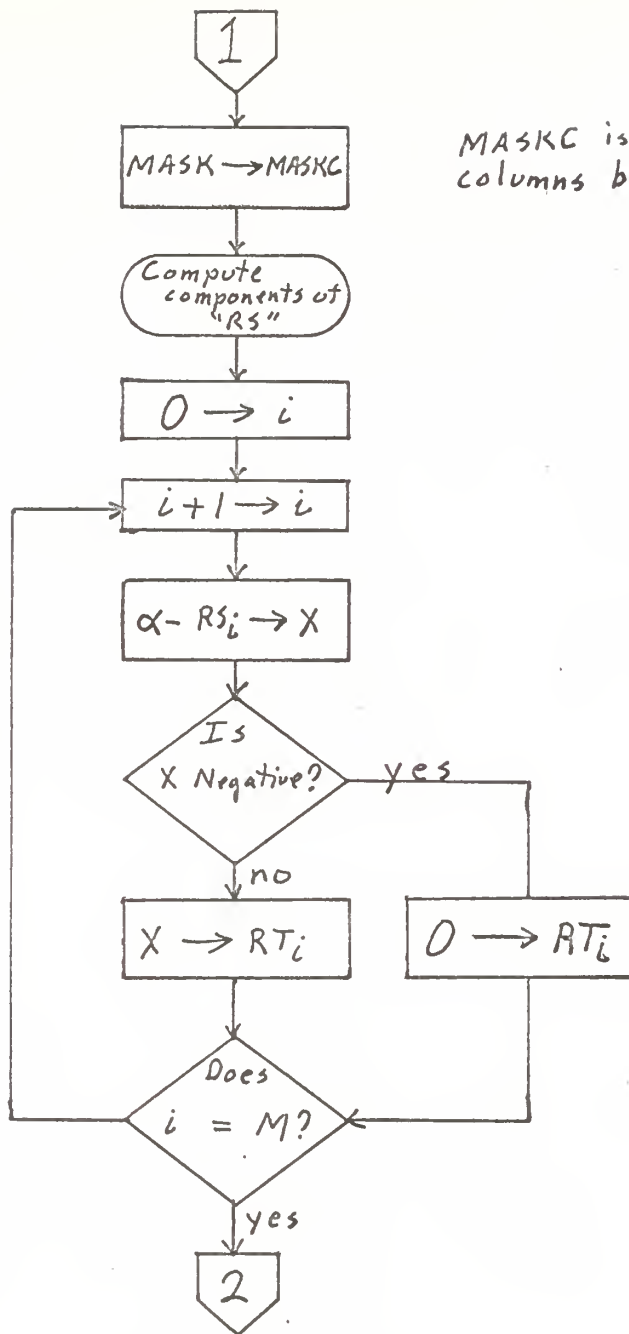
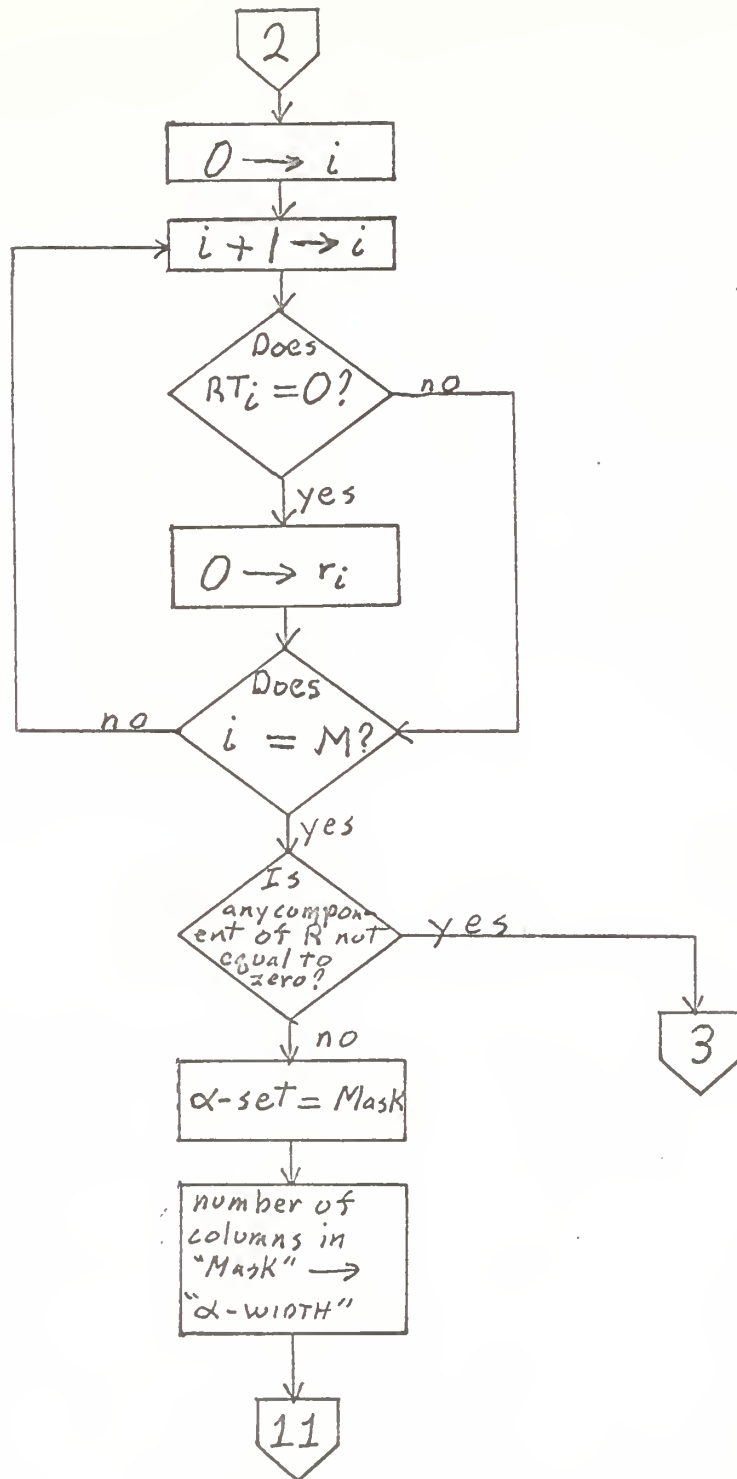
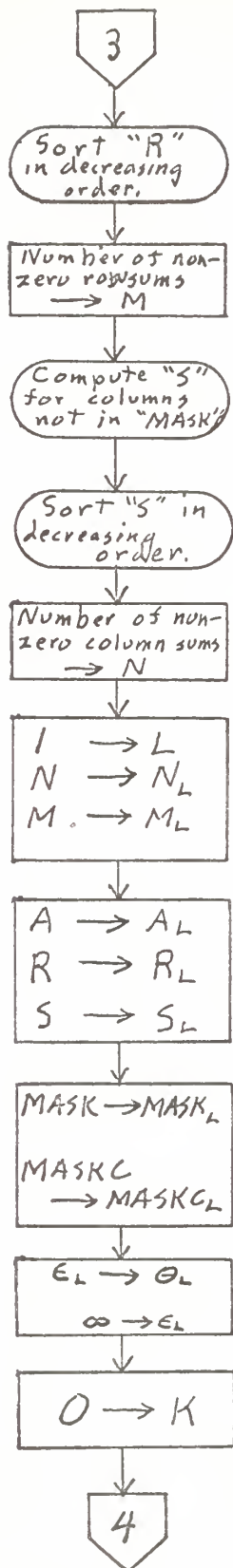


Figure 21

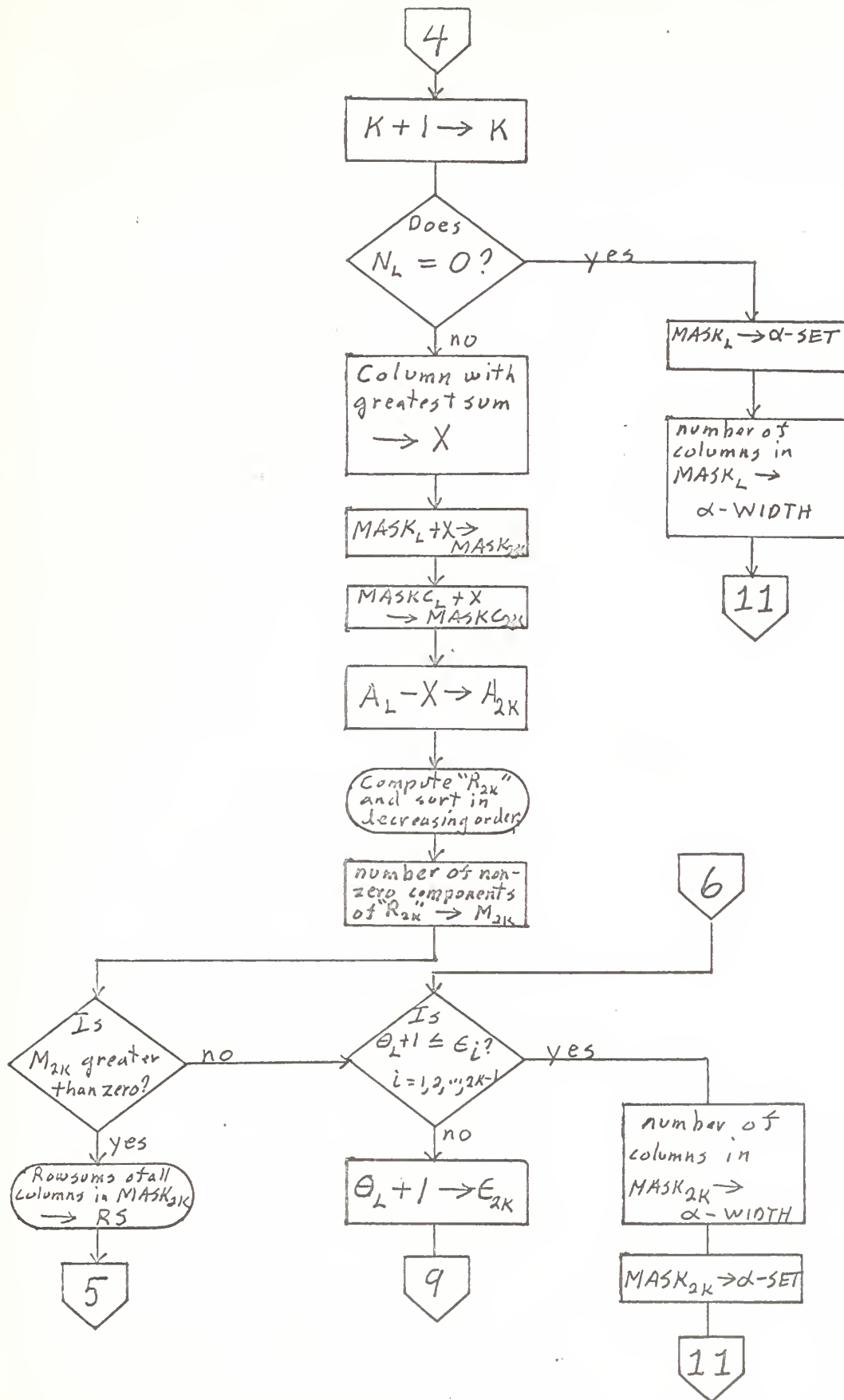


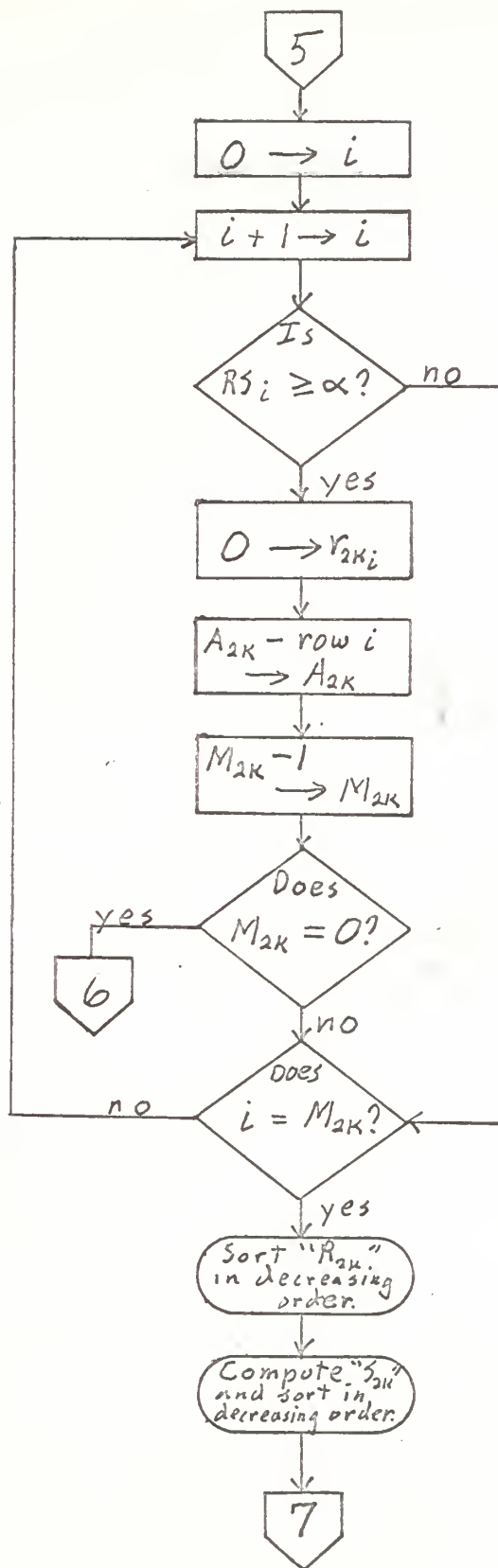
MASKC is the list of columns branched upon.

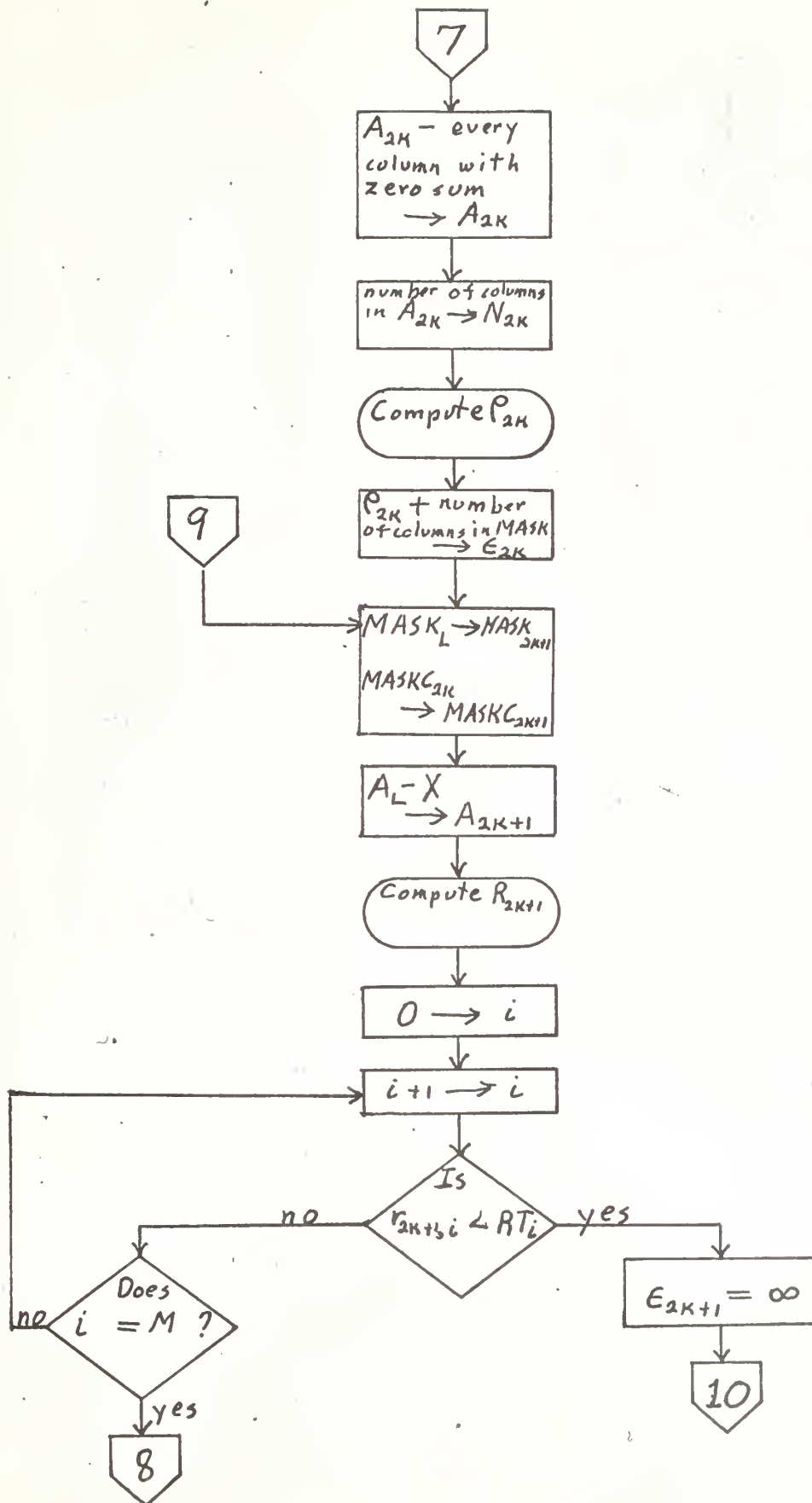


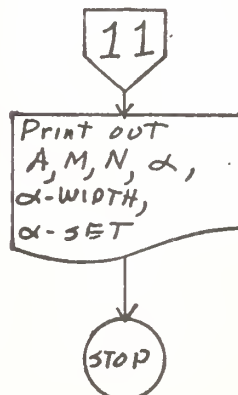
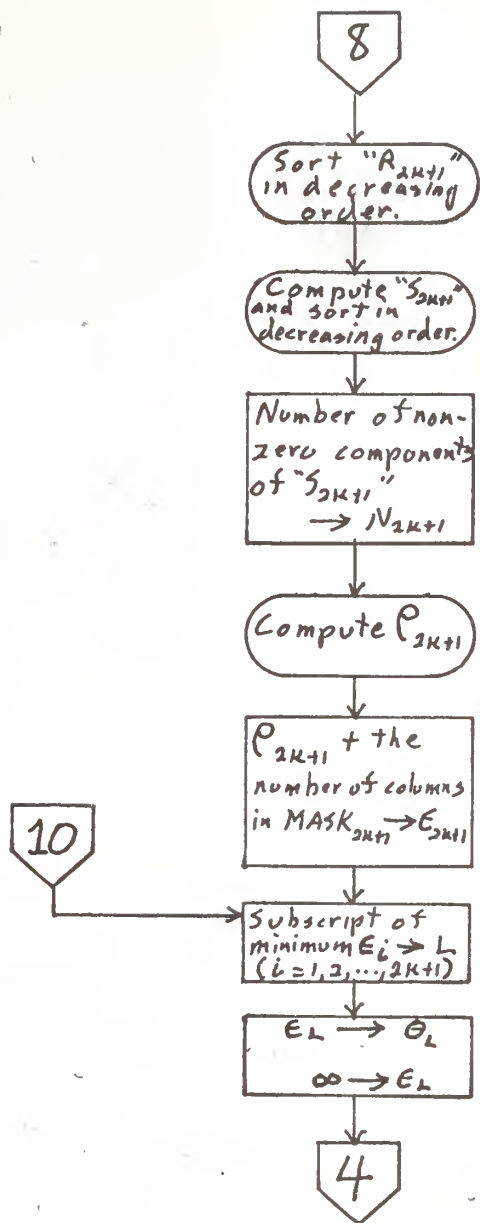


L is the index of the branching matrix









the assembly language level; but we avoid this by taking advantage of a capability of FORTRAN 63 which permits the programmer to define his own type of arithmetic.

The CDC 1604 word size is 48 bits. We chose to write the program to accept matrices up to dimensions 144 by 144. We store a single row of the matrix in three consecutive computer words; hence an entire matrix requires only 432 words of storage. The first word of row i contains elements a_{i1} through a_{i48} ; the second word contains a_{i49} through a_{i96} ; and the third word contains a_{i97} through a_{i144} .

We define, according to the rules of FORTRAN 63, a TYPE LOGIC5 arithmetic in which an elemental word consists of three consecutive words of memory. We call such an elemental word a TYPE LOGIC5 word. Thus, one LOGIC5 word is equivalent to one entire row of a matrix, or other variable which needs to be three computer words in length. For instance, we shall require several masks with which to derive the various sub-matrices, and each such mask must consist of three computer words.

We shall need to take logical sums and products, to complement words, to clear words of ones in certain bit positions; and we shall require a method of generating a 1 in any of the 144 bit positions of a LOGIC5 word. We define, through the subroutine, Q1QMATH, the symbol "+" to mean logical sum; the symbol "*" to mean logical product; the symbol "-" with two arguments to mean "set the i^{th} bit of the first argument to zero if the i^{th} bit of the second argument is one"; and the symbol "~" with one argument to mean "complement the argument". We also define "ARGUMENT /j" to mean "set the j^{th} bit of the argument to one, and all other bits to zero, counting from the leftmost bit position of the argument".

The only requirement we have for generating the sub-matrices of the given matrix is that we must be able to compute the corresponding row sums and column sums for use in estimating $\tilde{\epsilon}$. We may compute the row and column sums in the computer program without deriving each of the sub-matrices through the use of suitable masks. We require two such masks; one is a mask of columns upon which the program has already branched; and the other is a mask of columns chosen for inclusion in the minimal α -set at the current branch. In each case, a 1 in the i^{th} bit position of a mask indicates that column i is a member of the set of columns which the mask represents.

Almost all arguments used in the various subroutines are stored in COMMON. This decreases the computation time at the expense of requiring difficult to follow indexing of the parameters. Most such parameters are stored in an array, IDATA. This array is really three consecutive arrays of parameters associated respectively with the matrices A_L , A_{2K} , and A_{2K+1} of Figure 2¹. The correspondence between IDATA and the mnemonic variable names may be found in the EQUIVALENCE statement near the beginning of the program.

The masks and bound of all matrices must be retained in storage, but other parameters, (row sums, column sums, dimensions, etc.) are recomputed each time they are required. If a random access storage device (such as a magnetic disc) is available a savings of computation time would result from the storage of these parameters.

Up to 2000 sets of parameters can be retained in core storage simultaneously. When this limit is reached, the section of the program from statements 192 to 193 searches for any sets of parameters no longer required, discards them and compresses the remaining parameters into the

A MATRIX OF DIMENSION (8) BY (9).

6000000000000000	0	0
3000000000000000	0	0
1400000000000000	0	0
6000000000000000	0	0
1400000000000000	0	0
1040000000000000	0	0
1020000000000000	0	0
3000000000000000	0	0

1-WIDTH OF THE MATRIX IS 4.

NUMBER OF ITERATIONS REQUIRED- 13
TIME FOR COMPUTATION- 0MIN, 1SEC.

COLUMNS IN MINIMAL REPRESENTATIVE SET ARE-
2, 4, 7, 8,

Figure 22

front of the storage area. This effectively increases storage space up to the point at which there are 2000 current branches of the algorithm. (Current branches are those branches for which the corresponding estimate of ϵ is less than "infinity".)

Sample output is shown in Figure 22. This matrix can be recognized as the incidence matrix of the communications network discussed in Section one. Note that the matrix is printed in octal format which must be converted by hand to the proper (0,1) form. Each digit of the output represents three elements of the matrix; for example, the digit "5" represents the three elements "1, 0, 1".

Short, but descriptive comments separate major sections of the program listing by tasks, and introduce each of the subroutines. The various CDC 1604 instruction manuals and programming manuals may be consulted for further information.

The program is not very efficient in its present form: many programming conveniences such as the use of TYPE LOGIC5 arithmetic, and the use of subroutines, makes writing of the program simpler at the expense of generating many otherwise unnecessary instructions. As a first step toward improving the efficiency, the author recommends elimination of TYPE LOGIC5 arithmetic, substituting in its place, CODAP1 subroutines to perform the necessary substitute operations, and using direct calls to these subroutines in place of the operations symbols. In addition, it is recommended that all present subroutines written in FORTRAN 63 be incorporated into the main program. Program space is not critical in a computer the size of the CDC 1604, and by writing the subroutines as part of the main program, advantage may be taken of task specialization. For instance, subroutine ROWSUM computes

the sum of all M rows of the matrix each time it is called. It takes as much time to compute a row sum which is zero as one which is not; but we have information which could be used to specialize the routine so that it skips over rows whose sum is zero.

A still better technique would be to write the entire program at the assembly language level; especially if the user intends to use the program for more than the solution of a few matrices.

6.1 Results of Using PROGRAM WIDTH.

If a matrix has α -width, ϵ , and we were to attempt to find the α -width by looking at all possible sets of α columns, then all possible sets of $\alpha + 1$ columns, and so forth up to all possible sets of $\epsilon - 1$ columns and finally some sets of ϵ columns, we should have to look at X sets of columns for

$$(17) \quad \sum_{k=\alpha}^{\epsilon-1} \binom{n}{k} + 1 \leq X \leq \sum_{i=\alpha}^{\epsilon} \binom{n}{i}$$

We should have to look at this number of sets of columns using the branch and bound algorithm also, if all of the estimates of ϵ which were current turned out to be equal. It is conceivable that this could happen for some problem; hence we must take (17) as an upper bound on the number of branches which must be investigated by the program. Now the branch and bound algorithm is not the most efficient way to search subsets of columns, so we are quite interested in determining just how far below the upper bound we can stay by branching and bounding.

Since we cannot express any theory to demonstrate the efficiency of the algorithm, the only choice open to us was to solve many problems of varying sizes in hopes that trends could be established. It is for this reason that subroutines RANDOM and RANDGEN were added to the program. These two subroutines generate matrices of any size up to 144

by 144. A uniform random number generator is used to generate three consecutive random numbers which represent one row of a matrix of 144 columns. If a matrix of N columns is desired ($N < 144$) bit positions $N + 1, \dots, 144$ of the three word element are set to zero. The number of ones remaining in the three words is computed and compared to a user supplied argument, NONES. If NONES is less than the remaining ones in the three word element, another set of three words is generated, the appropriate bit positions cleared to zero, and then the logical product of the two elements is taken. This procedure is repeated until NONES is greater than or equal to the number of ones remaining in the three word element. Thus NONES represents the maximum permissible row sum of any row in the matrix. The three word element is then assigned as a row of the matrix, and the procedure is repeated until an entire matrix has been generated. We are thus reasonably sure of a random distribution of ones throughout the matrix, and we have some control over the density of ones in the matrix. Matrices of any dimension are generated in no more than a few seconds.

Our original plan was to generate and solve five matrices of each of 112 sets of dimensions for the matrix. It was felt that such a set of matrices would be a statistically significant sample from which computation time could be functionally related to such parameters as matrix dimensions. Unfortunately, time has prevented the completion of this scheme. Hence all remarks that follow in this section are without statistical significance.

We have been able to generate and solve over 200 matrices of varying dimensions for their 1-width; one being by far the most important

value for α . Dimensions of matrices generated were from the 8 by 9 problem of section one to matrices of dimensions 144 by 25, and 35 by 100. Some relatively square matrices of size 50 by 45 are included. As is to be expected, computation time varies directly with number of branches considered when matrix dimensions are held constant. Let us therefore make some remarks about the number of branches considered by the program in solving these matrices.

It is clear that the number of branches is a function of the number of columns and of the actual 1-width of the matrix. Not quite so obvious is that the number is a function of the number of rows in the matrix. However the fluctuations apparent in the number of branches is very wide. For instance, for one matrix 260 branches were taken while for another, 1536 were taken. Both matrices were of dimensions 35 by 35, and had a 1-width of seven. It is apparent that other factors must be involved. One such factor is the distribution of ones in the matrix. One matrix, a Steiner triple system [5], which is a matrix which among other properties has all row sums equal and all column sums equal; required investigation of 1216 branches before computing the 1-width as nine. Yet this matrix had only 35 rows and 15 columns. The symmetry of the matrix made it difficult to weed out unprofitable branches. Another matrix of dimensions 50 by 45 exceeded the capacity of the program storage after 2184 branches. In every case, however, the number of branches were below the upper limit given by (17). Values seldom exceeded 600 for any of the matrices.

Of more practical interest is the time required for computation. The CDC 1604 has an effective cycle time of 4.8 μ sec. The longest

time required to solve any problem was 36 minutes, although there were problems which had not been solved when the program was stopped by the operator after about 45 minutes. The matrix which required 36 minutes was of dimensions 50 by 45 and had a l-width of six. The relationship between branches and computation time is rather interesting. Matrices of dimensions 125 by 25 required a little over one second per branch whereas matrices of dimensions 25 by 120 required between two and three seconds per branch. This seems to verify that advantage could be gained by eliminating the LOGIC5 arithmetic in favor of more efficient methods, since the amount of LOGIC5 arithmetic required increases with number of columns.

Computation time was graphed on semi-log paper versus 1) number of columns, 2) number of rows, and 3) l-width of the matrix. Figure 23 is a graph of time versus number of columns for matrices of 25 and 35 rows. Figure 24 is of time versus number of rows for matrices of 25 and 35 columns, and Figure 25 is of time versus l-width for matrices of dimensions 20 by 20.

From these graphs, it seems reasonable to conclude an exponential increase in computation time versus both number of rows and number of columns. No hypothesis is made about the parameters of the function. Our method of generating matrices degrades the validity of Figure 25. In order to create matrices of high l-width, we can only lower the density of ones in the matrix. This, in turn, increases the likelihood of rows of sum one; which results in an artificial simplification of the problem. This is apparent especially in the case of the matrices of l-widths nine and ten in Figure 25.

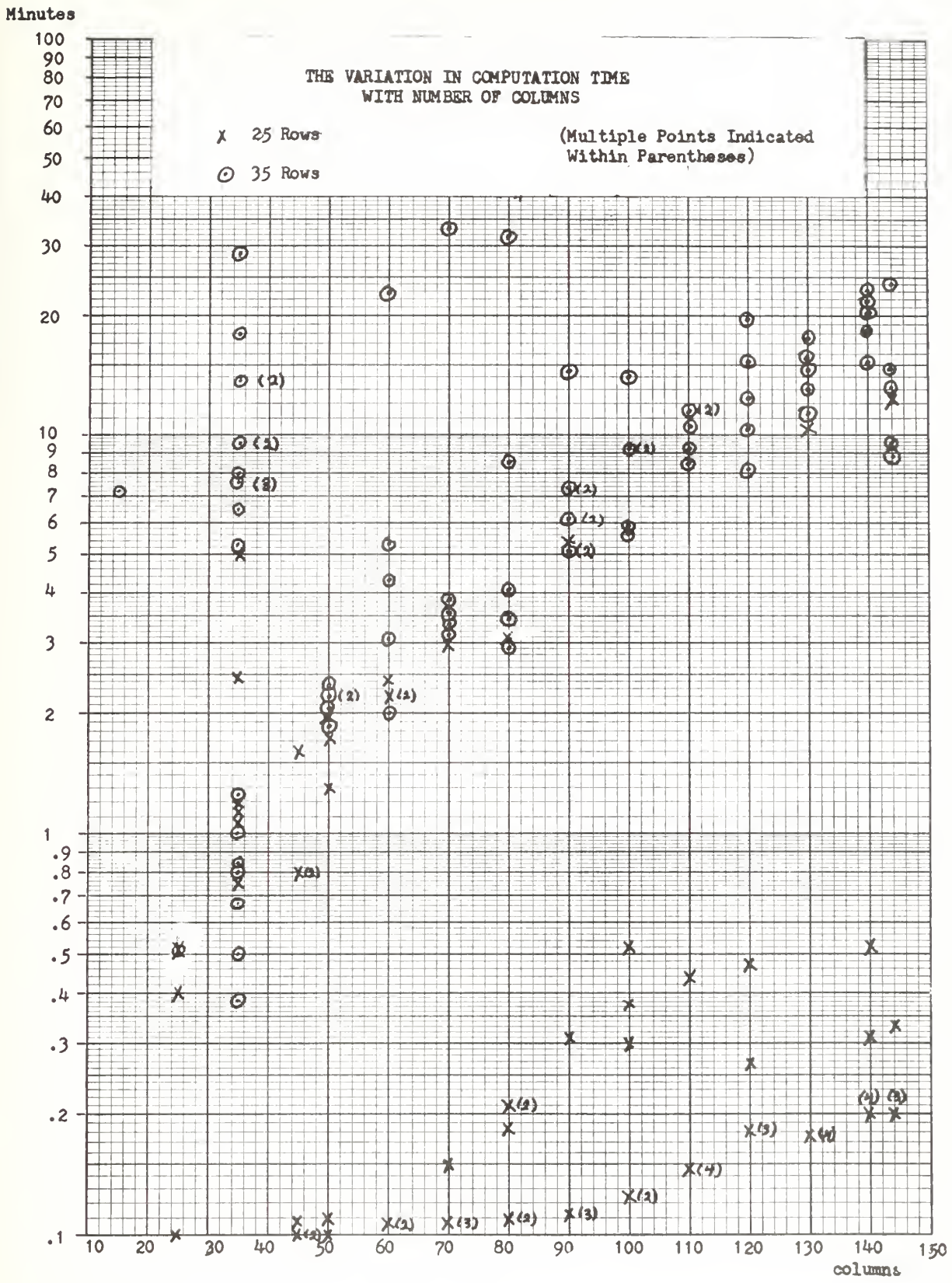


Figure 23

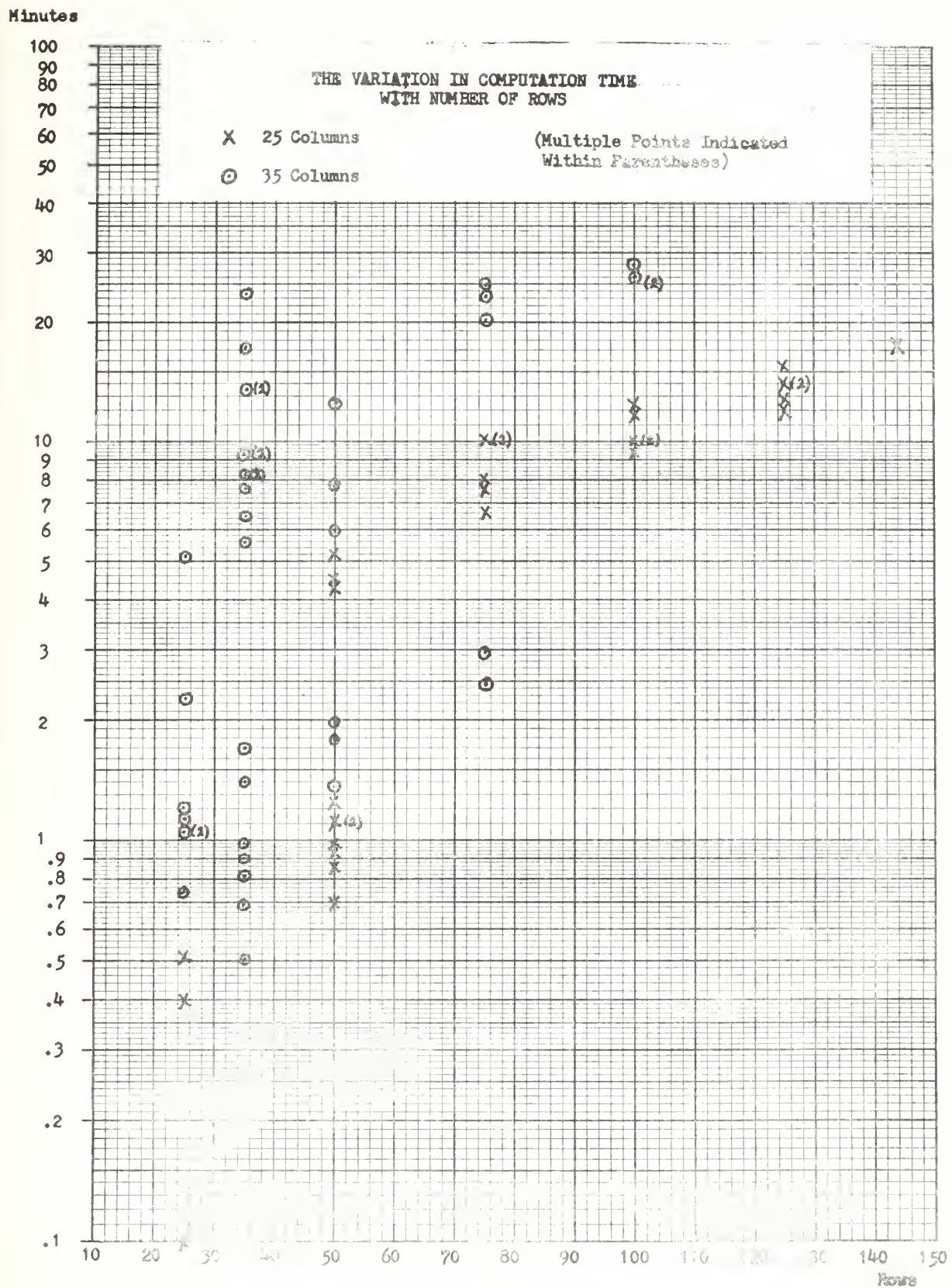


Figure 24

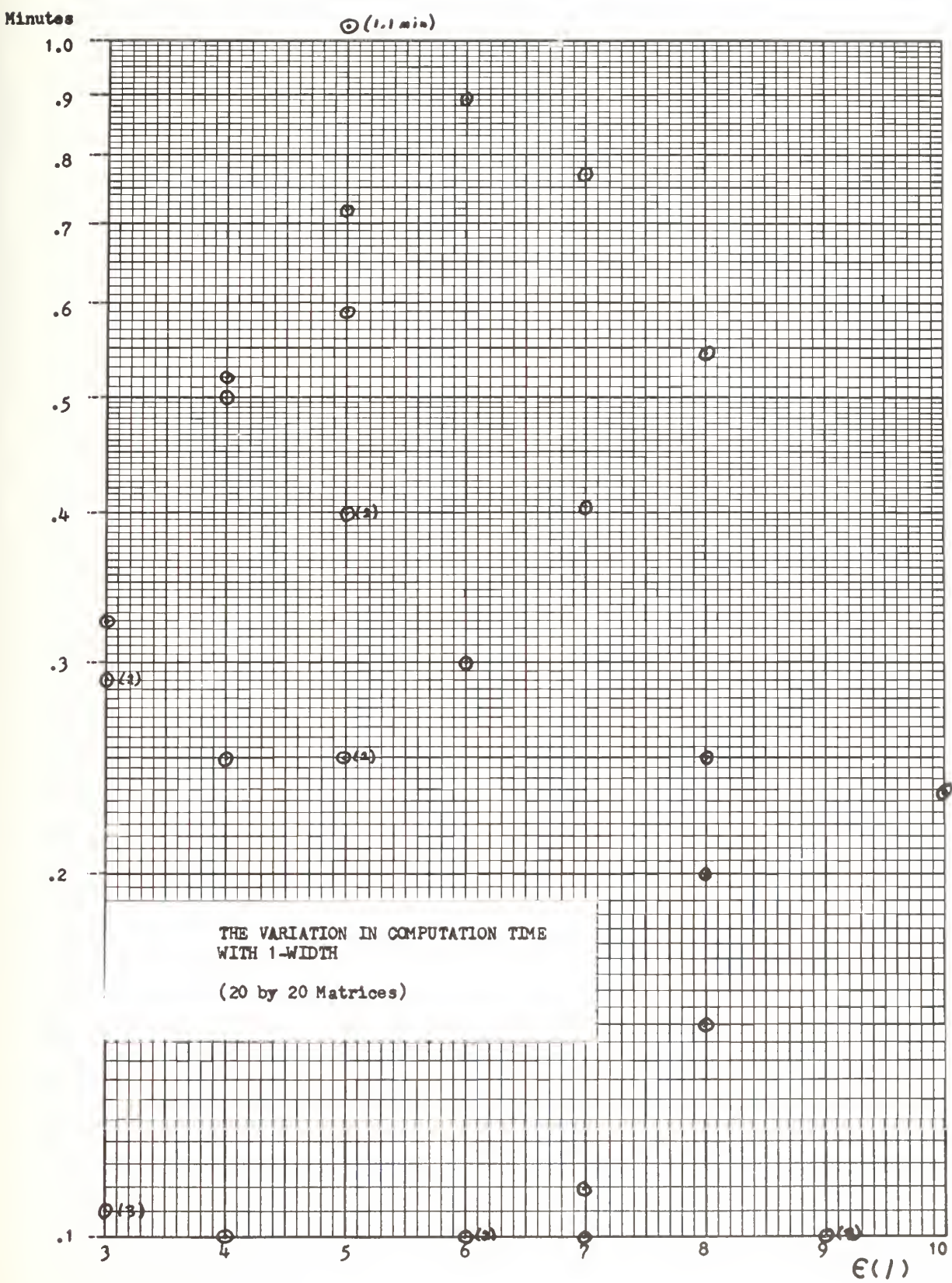


Figure 25

The validity and usefulness of the algorithm has been established by the above results. Most computation times were under 20 minutes, and it is felt that computation times could be reduced more than 25% by cleaning up the program and doing without the programming convenience of TYPE LOGIC5 arithmetic.

7. Applications and Extensions.

In this final section we consider applications of the branch and bound algorithm to solution of real-world problems, and certain famous problems of the mathematical puzzle category. We shall also propose certain extensions of the algorithm as presented in section three, which enlarge the class of problems which may be solved. Certain of the problems may, indeed, be more easily solved by other methods, but they are presented here to illustrate the variety of problems which may be formulated in terms of finding the C-width of a $(0,1)$ matrix.

7.1 The Eight Queens Problem.

A famous mathematical puzzle is the following: place the maximum number of queens on a chess board so that no two may attack each other. We construct a graph of 64 nodes, one for each square on the chessboard. Connect two nodes if a queen may move from one node to the other. The minimal 1-set of the node-arc incidence matrix is a minimal set of nodes that touch all arcs of the graph. Now since this matrix has row sums which are all equal to two, the sub-matrix consisting of all columns not in the minimal 1-set has row sums of at most one. Hence in the graph corresponding to this sub-matrix, there is no connection between any of the nodes. Thus the complement of the minimal 1-set of nodes

represents square at which the maximum number of queens may be placed so that no two may attack each other. This problem is a special case of a class of problems which also includes the targeting problem of Section one. We next present a description of this general class of problem.

7.2 The Connecting Nodes Problem.

Find the fewest number of nodes that touch all arcs in a graph. Here the rows of the incidence matrix are arcs of the graph, and the columns are nodes. The 1-width of the incidence matrix is the solution to the problem. In addition to the targeting problem of Section one, another problem of this type is the following:

Given a communications system of some type (let us say a system of highways connecting towns), what is the minimum number of arcs (highways) which must be kept safe from attack (natural disaster, etc.) so that no node is isolated? In order to solve this problem, we construct an incidence matrix as follows: list the arcs as columns and the nodes as rows. Let $a_{ij} = 1$ if the i^{th} node is a terminal of the j^{th} arc. Then the 1-width of this matrix is the solution to the problem.

7.3 Simplification of Logical Functions.

We present this problem to show the range of interpretations which may be made from the concept of α -width. The reader is cautioned, however, that the incidence matrix required for using the branch and bound algorithm to solve this problem is likely to become prohibitively large.

Given a truth table for a proposition letter formula, F , in r proposition letters, $p_1, \dots, p_r$; find a disjunctive normal form for F which has the fewest number of terms. If we let "&" represent the conjunction operator; "+" represent the disjunction operator; and $\bar{a}$ be the negation of a , then the disjunctive normal form for a proposition letter formula is of the form:

$$(p_1 \& p_2) + (p_1 \& p_3 \& p_4) + (p_3 \& p_4) + \dots$$

where the p_i are proposition letters. The expressions enclosed within parentheses are called terms. The problem is of interest in switching circuits and in the logical design of digital computers.

For columns of the incidence matrix take all terms having one of the forms: q_i ; $q_i \& q_j$; $\dots$; $q_1 \& \dots \& q_r$; where q_i is either p_i , or its negation; and such that the term takes the value "true" only if F does also for all values of the p_i not explicitly present in the term. For example, if $p_1 \& \bar{p}_3$ is a term of F in three proposition letters, p_1, p_2, p_3 ; then both $F(p_1, p_2, \bar{p}_3)$ and $F(p_1, \bar{p}_2, \bar{p}_3)$ must be true if $p_1 \& \bar{p}_3$ is true. We next construct a row of the incidence matrix for each "true" entry of F in the truth table. Place a one in the column corresponding to the assignment of values to $p_1, \dots, p_r$ which makes up the entry in the truth table corresponding to the "true" entry of F . Then place ones in all other columns which are also true for this assignment of values to the p_i . Thus if $p_1 \& p_2 \& p_3$ makes F true, a row of the matrix would have a one under this column label as well as under $p_1 \& p_2$; $p_1 \& p_3$ and so forth.

As an example, consider the truth table of Figure 26. The columns of the matrix would be labelled " $p_1 \& \bar{p}_2$ "; " $p_1 \& \bar{p}_3$ "; " $p_2 \& \bar{p}_3$ "; " $p_1 \& p_2 \& \bar{p}_3$ "; " $\bar{p}_1 \& p_2 \& \bar{p}_3$ "; " $p_1 \& \bar{p}_2 \& \bar{p}_3$ "; " $p_1 \& \bar{p}_2 \& p_3$ ". The four rows would have ones in columns labelled as follows:

p_1	p_2	p_3	F
f	f	f	f
f	f	t	f
f	t	f	t
f	t	t	f
t	f	f	t
t	f	t	t
t	t	f	t
t	t	t	f

Figure 26

Row 1: $p_2 \& \overline{p_3}; p_1 \& \overline{p_2} \& \overline{p_3}$

Row 2: $p_1 \& \overline{p_2}; p_1 \& \overline{p_3}; p_1 \& \overline{p_2} \& \overline{p_3}$

Row 3: $p_1 \& \overline{p_2}; p_1 \& \overline{p_2} \& p_3$

Row 4: $p_1 \& \overline{p_3}; p_2 \& \overline{p_3}; p_1 \& p_2 \& \overline{p_3}$

Clearly, $p_2 \& \overline{p_3}$ and $p_1 \& \overline{p_2}$ are a minimal 1-set of representatives for the matrix, and $F = (p_1 \& \overline{p_2}) + (p_2 \& \overline{p_3})$ is a minimal disjunctive normal form.

7.4 The Minimal C-cover Problem.

It would be quite simple to extend the computer program to solve the minimal C-width problem. Essentially all that would be necessary is input revision to accept the vector, C, and the initial setting of the vector, RT to C. Of course, $\tilde{\epsilon}$ could not be calculated, but would have to be estimated using exactly the same subroutine which is presently used for the situation, $\alpha > 1$. For an entirely different algorithm for solving the minimal C-cover problem (and hence also the α -width problem) see [8]. The minimal C-cover extension is of interest primarily as a first step to a more involved and more useful extension.

7.5 A Minimal Cost C-cover.

One of the more obvious deficiencies of the solution to the targeting problem of section one is that when only one α -set is computed, that particular set might include a target very heavily fortified whereas one not as heavily fortified might have been a member of another minimal α -set. One approach to remedy this deficiency would be to compute all minimal α -sets; and indeed the approach will be mentioned subsequently. However, it is also possible that for some variety of reasons, it would be preferable even to destroy more than the minimum number of targets. The term, preferable, indicates that there might be a utility function or cost function associated with the problem.

The extension of the algorithm so that it may handle costs associated with the columns is perhaps the most interesting extension that we shall discuss. The author believes that this extension might result in a decrease in the computation time required. The belief is based upon the observation that the lower bounds calculated in the present program are relatively close to each other. Thus there is entirely too much switching away from one branch, to another, and then back to the original branch. With a wide difference among the column costs, however, the differences among the various estimates of the C-width of the original matrix should be equally wide. This will serve to reduce the unnecessary switching from branch to branch. That is, it is more likely that when a branch is dropped by the algorithm, it is because that branch has become unprofitable.

Suppose we assign to each column of the matrix a cost, which need not be integral, and may be positive, negative, or zero. We would then be interested in finding a C-cover (or an α -set) which has

minimum cost associated with it. Of course, such a C-cover might not be a minimal C-cover as defined previously.

The modification to the computer program would be surprisingly simple. The cost vector would be read in, and let us assume stored in COMMON. Now the "infinity" for unfeasible column combinations must be increased to some arbitrarily large number. An estimate of $\epsilon(C)$ would be calculated for each sub-matrix using the same subroutine as presently used for $\alpha > 1$. From subroutine BOUND, however, the program would enter a new subroutine, such as subroutine COST presented in Figure 27. In this subroutine, a cost for the p just computed would be estimated. The estimate would be optimistic in the sense that the cost for the columns would be the sum of the smallest cost components not already used on this branch. For example, consider a cost vector, (1,2,3). Assume that, on the current branch, column one has been either included or excluded, and that we have computed $p = 1$. Then the cost for the sub-matrix would be estimated as two; and the estimate of the cost for the minimal C-cover would be two plus the cost of column one, if column one had been included, or two, if column one had been excluded.

Finally, either in the same subroutine, or in the main program, the cost of the set of currently included columns would be computed and stored in place of the argument, VCOL(I) of the current program. Also, VEPSILON(I) of the current program would be replaced by the sum of VCOL(I) and the cost estimate just computed in subroutine COST, as described in the above paragraph.

The reader is reminded that the subroutine COST of Figure 27 has not been checked out and that the remarks in this section about decreasing total computation time represent merely the author's intuition and are not based upon observations.


```

SUBROUTINE COST (EPSILON, I)
COMMON/BLOCKH/CCOST(144)/BLOCKB/IDATA(1761)
EQUIVALENCE (MASKC,IMASK(3))
TYPE LOGIC5 (3) MASKC, BIT
DIMENSIONS TEMP(144)
DO 5 J = 1, 3
5 IMASK(J) = IDATA((587*(I-1))+582+J)
N = IDATA((587*(I-1))+3)
BIT = BIT * MASKC
IF (BIT.EQ.0) 10, 20
10 K = K + 1
TEMP(K) = CCOST(J)
IF (TEMP(K).LT.TEST) 15, 20
15 TEST = TEMP(K)
K1 = K
L = EPSILON
20 CONTINUE
IF (K.LT.L) 25, 30
25 EPSILON = 1.E+20
RETURN
30 EPSILON = TEST
J = 1
35 J = J + 1
TEST1 = TEMP(1)
DO 45 M = 2, K
IF (TEMP(K).GE.TEST.AND.TEMP(M).LT.TEST1.AND.K1.NE.M) 40, 45
40 TEST1 = TEMP(M)
45 CONTINUE
EPSILON = EPSILON + TEST1
TEST = TEST1
IF (J.EQ.L) 50, 35
50 RETURN
END

```

Figure 27

Once both of the above extensions have been programmed we may use the algorithm to solve a large variety of problems which are a subclass of the set of integer programming problems.

7.6 An Integer Program with a (0,1) Constraint Matrix.

We merely point out in this section, a formulation of a problem which the extended algorithm can solve. Given the system of linear inequalities:

$$\sum_{j=1}^n a_{ij} \cdot x_j \cong b_i \quad i = 1, 2, \dots, m$$

where a_{ij} is either zero or one; find values for each x_j such that x_j is either zero or one, which minimizes:

$$\sum_{j=1}^n c_j \cdot x_j.$$

Here the c_j are costs, and the b_i are the components of what has previously been called the C vector.

7.7 Constraints on Combinations of Columns.

Suppose that upon any of the problems which may be solved by extensions of the algorithm, we impose constraints of the following type: if column a is included in the C-cover, then column b must be excluded.

We could write a relatively short subroutine to handle this type of constraint. It would be necessary to put the constraints in a convenient form, say for each constraint construct a mask of zeros except in the bit positions corresponding to columns which cannot be included together. For example, let there be five columns and assume two constraints: that columns one and two cannot be included together, and that columns three and five cannot be included together. Then the two masks would be:

We put these constraints into the program in some convenient fashion (probably by the same system used for putting the A matrix in the current program); and write a subroutine to compute the logical product of each constraint with the mask of included columns. If there are no ones in the product for any constraint, the subroutine must set a "current" cost vector equal to the input cost vector. If there are two or more ones in any single product, the subroutine must indicate that an infeasible column selection has been made. Finally, for each constraint with exactly one "1" in the product, set the "current" cost of every "1" in the constraint to "infinity"; except of course, the "1" representing the current column inclusion. We use the "current" cost vector in computing bounds instead of the input cost vector.

7.8 Finding All Minimal C-covers of the Matrix.

It is possible to simplify the search for minimal C-covers once the first one has been located. The same algorithm applies except that we have information to rule out as infeasible, any set of columns which yields an estimate of $\epsilon(C)$ larger than the computed C-width. Although the simplification would contribute to a substantial savings in computation time for each additional minimal C-cover, it is believed that for most problems the search for all minimal C-covers would require more computation time than the results would warrant.

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APPENDIX I

Listing of PROGRAM WIDTH


```

PROGRAM WIDTH
TYPE INTEGER A5, ALPHA, COLL, COL2, COL3, COLSET, EPSILON,
1 EPSILON1, EPSILON2, EPSILON3, R, R1, R2, R3, RO1, RO2, RO3, S1, S2, S3, SO1, SO2, SO3, T, VEPSILON, SUM, RT, VCOL
2 S2, S3, SO1, SO2, SO3, T, VEPSILON, SUM, RT, VCOL
TYPE LOGIC5 (3) A, BIT, COLUMNS, MASK1, MASK2, MASK3, MASKC1,
1 MASKC2, MASKC3, ROW, MASKR, VMASK, VMASKC
COMMON/BLOCKA/A(144)/BLOCKB/IDATA(1761)/BLOCKC/MO,NO,ALPHA,EPSILON,WID00001
,RT(144)/BLOCKD/VEPSILON(2000)/BLOCKE/ROW,SUM/BLOCKG/IRAND
1 DIMENSION A5(3,144), COLSET(144), R1(144),R2(144),
WID00002
WID00003
WID00004
WID00005
WID00006
WID00007
WID00008
WID00009
WID00010
WID00011
WID00012
WID00013
WID00014
WID00015
WID00016
WID00017
WID00018
WID00019
WID00020
WID00021
WID00022
WID00023
WID00024
WID00025
WID00026
WID00027
WID00028
WID00029
WID00030
WID00031
WID00032
WID00033
WID00034
WID00035
WID00036
1 R3(144), RO1(144),RO2(144), RO3(144), S1(144), S2(144),
2 S3(144), SO1(144), SO2(144), SO3(144), IDMP(587) , ITEM(2000),
3 VMASK(2000), VMASKC(2000), VCOL(2000)
EQUIVALENCE (A5,A), (IDATA(1),I1), (IDATA(2),M1), (IDATA(3),N1),
1 (IDATA(4),R1), (IDATA(148),RO1), (IDATA(292),S1), (IDATA(436),
2 SO1), (IDATA(580),MASK1), (IDATA(583),MASKC1), (IDATA(586),COL1
3 ),(IDATA(587),EPSILON1), (IDATA(588),I2), (IDATA(589),M2),
4 (IDATA(590),N2), (IDATA(591),R2), (IDATA(735),RO2), (IDATA(879)
5 ,S2), (IDATA(1023),SO2), (IDATA(1167),MASK2), (IDATA(1170),
6 MASKC2), (IDATA(1173),COL2), (IDATA(1174),EPSILON2),
7 (IDATA(1175),I3), (IDATA(1176),M3), (IDATA(1177),N3), (IDATA
8 (1178),R3), (IDATA(1322),RO3), (IDATA(1466),S3), (IDATA(1610),
9 SO3), (IDATA(1754),MASK3), (IDATA(1757),MASKC3), (IDATA(1760),
* COL3), (IDATA(1761),EPSILON3)
READ IN INITIAL OCTAL RANDOM NUMBER. SET ALL DATA WORDS TO ZERO
READ (4,900) IRAND
5 DO 10 I = 1,1761
10 IDATA(I) = 0
DO 15 I = 1, 144 $ RT(I) = 0
15 COLSET(I)=0
DO 20 I = 1, 500 $ VMASK(I) = VMASKC(I) = VCOL(I) = 0
20 VEPSILON(I) = 0

```


WID000037
WID000038
WID000039
WID000040
WID000041
WID000042
WID000043
WID000044
WID000045
WID000046
WID000047
WID000048
WID000049
WID000050
WID000051
WID000052
WID000053
WID000054
WID000055
WID000056
WID000057
WID000058
WID000059
WID000060
WID000061
WID000062
WID000063
WID000064
WID000065
WID000066
WID000067
WID000068
WID000069
WID000070
WID000071
WID000072

```

C      INPUT DIMENSIONS OF MATRIX,(MAXIMUM 144 BY 144), ALPHA, AND IF
C      MATRIX IS TO BE RANDOMLY GENERATED, THE MAXIMUM NUMBER OF ONES
C      IN A ROW. THIS NUMBER MUST BE GREATER THAN ALPHA. IF ALPHA IS
C      NEGATIVE MATRIX WILL BE GENERATED RANDOMLY. IF POSITIVE, MATRIX
C      WILL BE READ IN FROM PUNCHED CARDS. IF ZERO, COMPUTER RUN
C      TERMINATES.
C
C      PREPARE MATRIX FOR INPUT ON PUNCHED CARDS BY CONVERTING ZFROS
C      AND ONES IN EACH ROW TO OCTAL DIGITS IN GROUPS OF THREE STARTING
C      FROM THE LEFT. PUNCH ONE CARD FOR EACH ROW, COLUMNS 1 THROUGH 48.
C      EXAMPLE--M = 15, A ROW OF THE MATRIX IS 0000110111110. PUNCH
C      THE CARD STARTING IN COLUMN ONE AS--03576.
C
25 READ (4,1000) MO,NO,ALPHA,NONES
   IF(ALPHA) 30, 40,45
30 ALPHA = -ALPHA
35 CALL RANDOM(MO,NO,ALPHA,NONES)
   GO TO 50
C
40 PRINT 1100
   STOP
C
45 READ (4,1200)((A5(I,J),I=1,3),J=1,MO)
C
C      ALL WORDS RESERVED FOR THE MATRIX BUT UNUSED IN THIS PROBLEM
C      ARE SET TO ZERO. MATRIX IS PRINTED OUT.
C
50 IF(NO.GT.48) 65, 55
55 DO 60 I=1,MO
60 A5(2,I) = A5(3,I) = 0 $ GO TO 80
65 IF(NO.GT.96) 80,70
70 DO 75 I=1,MO
75 A5(3,I)=0
80 IF(MO.LT.144) 85,95
85 I= MO+1 $ DO 90 J=I,144

```



```

90 A5(1,J) = A5(2,J) = A5(3,J) = 0
C
95 PRINT 1300, MO, NO, ((A5(I,J), I=1,3), J=1, MO)
C
C   START COMPUTATION OF ALPHA-WIDTH
C
    CALL TIME (ITIME1) $ I4 = IO = I1 = 1 $ JO = 1
    DO 105 I = 1, MO
    105 RO1(I) = I
    110 CALL ROWSUM(1)
    GO TO (115,120,120,120) I4
C
C   CHECK FOR PROBLEM FEASIBILITY
C   LOCATE ALL COLUMNS WHICH MUST BE IN EVERY ALPHA-SET
C
    115 DO 1154 I = 1, MO
    IF (R1(RO1(I)) - ALPHA) 140,1151,1154
    1151 DO 1153 J = 1, NO
    BIT = BIT/J
    IF (BIT*A(RO1(I)).EQ.0) 1153,1152
    1152 MASK1 = MASK1 + BIT $ MASKC1 = MASKC1 + BIT
    1153 CONTINUE
    1154 CONTINUE $ I4 = 2 $ ROW = MASK1 $ CALL SHIFTSUM
    VCOL(1) = SUM $ GO TO 110
C
C   COMPUTE RT OF THE BRANCHING MATRIX.
C
    120 DO 1201 I = 1, MO
    ROW = MASK1*A(I)
    CALL SHIFTSUM $ IDIFF = ALPHA - SUM
    RT(I) = XMAXOF(IDIFF,0)
    1201 CONTINUE $ GO TO (125,125,125,160) I4
C
C   MASK OUT ROWS SUFFICIENTLY REPRESENTED
C
    125 I = 0

```

WID000073
 WID000074
 WID000075
 WID000076
 WID000077
 WID000078
 WID000079
 WID000080
 WID000081
 WID000082
 WID000083
 WID000084
 WID000085
 WID000086
 WID000087
 WID000088
 WID000089
 WID000090
 WID000091
 WID000092
 WID000093
 WID000094
 WID000095
 WID000096
 WID000097
 WID000098
 WID000099
 WID000100
 WID000101
 WID000102
 WID000103
 WID000104
 WID000105
 WID000106
 WID000107
 WID000108


```

1251 I = I + 1 $ IF (R1(ROI(I)).GT.0.AND.RT(ROI(I)).LE.0) 1252,1254
1252 R1(ROI(I)) = 0 $ M1 = M1 - 1 $ DO 1253 J = 1, M1
1253 R1(J) = ROI(J+1) $ ROI(M1+1) = 0 $ I = I - 1
1254 IF (I.LT.M1) 1251,1255
C
C TEST FOR TERMINATION. IF TEST FAILS COMPUTE REMAINDER OF
C MATRIX PARAMETERS
C
1255 IF (M1.EQ.0) 130,135
130 COLUMNS = MASK1 $ GO TO 240
135 CALL BIGSORT(1)
CALL COLSUM(1)
CALL BIGSORT(2)
GO TO (155,155,160,160) I4
140 PRINT 1400, ALPHA $ GO TO 5
155 VMASK(1) = MASK1 $ VMASKC(1) = MASKC1
EPSILON1 = VEPSILON(1) = 2*NO
C
C PARAMETERS ENDING IN 1 REFER TO BRANCHING MATRIX. THOSE ENDING
C IN 2 REFER TO SUB-MATRIX WITH COLUMN INCLUDED IN ALPHA-SET
C THOSE ENDING IN 3 REFER TO SUB-MATRIX WITH COLUMN EXCLUDED
C FROM ALPHA-SET.
C
160 DO 161 I = 588, 1761
161 IDATA(I) = 0 $ IO = 12 = IO + 1 $ JO = JO + 1
C
C TEST FOR COMPLETION
C
IF (N1.EQ.0) 166,167
166 COLUMNS = MASK1 $ GO TO 240
C
C DERIVE SUB-MATRIX WITH BRANCHING COLUMN INCLUDED IN ALPHA-SET
C
167 RIT = BIT / SOL(1)
VMASK(JO) = MASK2 = MASK1 + BIT

```

WID00109
WID00110
WID00111
WID00112
WID00113
WID00114
WID00115
WID00116
WID00117
WID00118
WID00119
WID00120
WID00121
WID00122
WID00123
WID00124
WID00125
WID00126
WID00127
WID00128
WID00129
WID00130
WID00131
WID00132
WID00133
WID00134
WID00135
WID00136
WID00137
WID00138
WID00139
WID00140
WID00141
WID00142
WID00143
WID00144

WID00145
WID00146
WID00147
WID00148
WID00149
WID00150
WID00151
WID00152
WID00153
WID00154
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WID00156
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WID00159
WID00160
WID00161
WID00162
WID00163
WID00164
WID00165
WID00166
WID00167
WID00168
WID00169
WID00170
WID00171
WID00172
WID00173
WID00174
WID00175
WID00176
WID00177
WID00178
WID00179
WID00180

```

VMASKC(JO) = MASKC2 = MASKC1 + BIT
CALL ROWSUM(2)
IF (M2.EQ.0) 169,174
169 IF (EPSILON1+1.LF.MIN(JO-1)) 170,172
170 COLUMNS = MASKC2 $ GO TO 240
172 EPSILON2=VEPSILON(JO)=EPSILON1+1
VCOL(JO) = COL2 = COL1 + 1 $ GO TO 192

C
174 I = 0
175 I = I + 1
ROW = MASK2*A(RO2(I))
CALL SHIFTSUM $ IF (SUM.GE.ALPHA) 180, 190
180 R2(RO2(I)) = 0 $ M2 = M2 - 1 $ IF (M2.EQ.0) 169, 184
184 DO 185 J = I, M2
185 R2(J) = R2(J+1) $ R2(M2+1) = 0 $ I = I - 1
190 IF (I.LT.M2) 175, 191
191 CONTINUE
CALL BIGSORT(3)
CALL COLSUM(2)
CALL BIGSORT(4)
CALL BOUND(2)

C
C IF ALLOWED MEMORY IS USED UP, ATTEMPT TO DISCARD DATA NO
C LONGER NEEDED. IF UNSUCCESSFUL, TERMINATE.
C
192 IF (JO.GE.2000) 1920, 193
1920 N8 = 2 * NO $ JO = 0
DO 1922 I = 1, 2000
IF (VEPSILON(I) .LT.N8) 1921, 1922
1921 JO = JO + 1 $ ITEMP(JO) = I
1922 CONTINUE $ IF (JO.EQ.2000) 1923, 1924
1923 PRINT 1700, IO $ GO TO 5
1924 DO 1925 I = 1, JO $ K = ITEMP(I)
VEPSILON(I) = VEPSILON(K) $ VMASK(I) = VMASK(K)
VMASKC(I) = VMASKC(K)
1925 VCOL(I) = VCOL(K) $ J1 = 2*((JO+1)/2)

```



```

      IF (J1.GT.JO) 1926, 193
      1926 VEPSILON(J1) = N8 $ JO = J1
      193 VCOL(JO) = COL2 = COL1 + 1 $ VEPSILON(JO)=EPSILON2=EPSILON+COL2
C
C      DERIVE SUB-MATRIX WITH BRANCHING COLUMN EXCLUDED FROM ALPHA-SET
C
      194 IO = I3 = IO + 1
      JO = JO + 1
      VMASK(JO) = MASK3 = MASK1
      VMASKC(JO) = MASKC3 = MASKC2
      VCOL(JO) = COL3 = COL1
      CALL ROWSUM(3)
      DO 200 I = 1,MO
      IF (R3(I) .LT. RT(I)) 195,200
      195 EPSILON3 = VEPSILON(JO) = 2*NO $ GO TO 205
      200 CONTINUE
      CALL BIGSORT(5)
      CALL COLSUM(3)
      CALL BIGSORT(6)
      CALL BOUND(3)
      EPSILON3 =VEPSILON(JO) = EPSILON + COL3
C
C      CHOOSE NEXT BRANCHING MATRIX
C
      205 T = MIN(JO) $ IF (T-JO+1) 210, 225, 230
      210 DO 215 I = 1, 587
      215 IDATA(I) = 0 $ I4 = 3
      MASK1 = VMASK(T) $ MASKC1 = VMASKC(T) $ COL1 = VCOL(T)$I1=T
      EPSILON1 = VEPSILON(T) $ VEPSILON(T) = 2*NO $ GO TO 100
C
      225 I = 2 $ GO TO 233
      230 I = 3
C
      233 DO 235 J = 1,587
      235 IDATA(J) = IDATA(J + 587*(I-1)) $ VEPSILON(T) = NO*2
      I4 = 4 $ GO TO 120

```

WID00181
 WID00182
 WID00183
 WID00184
 WID00185
 WID00186
 WID00187
 WID00188
 WID00189
 WID00190
 WID00191
 WID00192
 WID00193
 WID00194
 WID00195
 WID00196
 WID00197
 WID00198
 WID00199
 WID00200
 WID00201
 WID00202
 WID00203
 WID00204
 WID00205
 WID00206
 WID00207
 WID00208
 WID00209
 WID00210
 WID00211
 WID00212
 WID00213
 WID00214
 WID00215
 WID00216


```

C      PREPARE OUTPUT
C
C      240 J = 0 $ DO 250 I = 1,NO $ BIT = BIT/I
          IF(BIT*-columns.EQ.0) 250,245
C      245 J = J + 1 $ COLSET(J) = I
C      250 CONTINUE
C
C      CALL TIME(itime2)
          itime = itime2 - itime1 $ itime1 = itime/60
          itime2 = itimef1/60
          itime = itime1 - itime2*60
C
          PRINT 1500, ALPHA,J,IO,itime2, itime
          PRINT 1600, (COLSET(K),K=1,J)
          GO TO 5
C
C      900 FORMAT (O16)
C      1000 FORMAT (2I3,I4,I3)
C      1100 FORMAT(13H END OF DATA.)
C      1200 FORMAT(3O16)
C      1300 FORMAT(1H1, 41X, 23HA MATRIX OF DIMENSION (,I3, 6H) BY (, I3,
          12H).////////(36X,3O16//))
C      1400 FORMAT(///39X,I3,38H-WIDTH DOES NOT EXIST FOR THIS MATRIX.)
C      1500 FORMAT(1H0,44X,I3,24H-WIDTH OF THE MATRIX IS ,I3,1H.,///
          131H NUMBER OF ITERATIONS REQUIRED-,I5,/23H TIME FOR COMPUTATION-
          212,4HMIN,I2,4HSEC.)
C      1600 FORMAT(// 43HOCOLUMNS IN MINIMAL REPRESENTATIVE SET ARE-,
          1//((15X,14(I3,1H,2X))//)
C      1700 FORMAT (28H0ALLOWED MEMORY EXCEEDED AT ,I5, 12H ITERATIONS.)
C
          END

```

WID00217
 WID00218
 WID00219
 WID00220
 WID00221
 WID00222
 WID00223
 WID00224
 WID00225
 WID00226
 WID00227
 WID00228
 WID00229
 WID00230
 WID00231
 WID00232
 WID00233
 WID00234
 WID00235
 WID00236
 WID00237
 WID00238
 WID00239
 WID00240
 WID00241
 WID00242
 WID00243
 WID00244
 WID00245
 WID00246
 WID00247
 WID00248


```

MIN000001
MIN000002
MIN000003
MIN000004
MIN000005
MIN000006
MIN000007
MIN000008
MIN000009
MIN000010
MIN000011
MIN000012
MIN000013

```

```

C
C
C
C
FUNCTION MIN(K)
  LOCATE THE SMALLEST COMPONENT OF A VARIABLE SIZE VECTOR
  AND RETURN ITS SUBSCRIPT
  COMMON/BLOCKD/IVEC(2000)
  MIN = 1 $ ITEST = IVEC(1)
  DO 10 J = 2,K
    IF(ITEST.LE.IVEC(J)) 10,5
  5 ITEST = IVEC(J) $ MIN = J
  10 CONTINUE
  RETURN
  END

```



```

C
C
C
C
SUBROUTINE ROWSUM(I)
C
C   COMPUTE THE LOGICAL PRODUCT OF A ROW OF THE MATRIX AND A MASK, AND
C   COMPUTE THE SUM OF THE RESULT
C
COMMON/BLOCKA/A(144)/BLOCKB/IDATA(1761)/BLOCKC/M,N,DUMMY(146)/
1  BLOCKE/ROW,SUM
  TYPE INTEGER SUM
  TYPE LOGIC5 (3) A, MASK, MASK1, ROW
  DIMENSION MASKI(3)
  EQUIVALENCE (MASKI,MASK)
C
  GO TO (5,10,15) I
5  DO 6 K = 1, 3
6  MASKI(K) = IDATA(582+K) $ J = 3 $ L1 = 2 $ GO TO 20
10 DO 11 K = 1, 3
11 MASKI(K) = IDATA(1169+K) $ J = 590 $ L1 = 589 $ GO TO 20
15 DO 16 K = 1, 3
16 MASKI(K) = IDATA(1756+K) $ J = 1177 $ L1 = 1176
20 MASK1 = -MASK $ JO = J + 144
  DO 30 K = 1, M
  GO TO (22,21,21) I
21 IF (IDATA(3+K).EQ.0) 30, 22
22 ROW = MASK1*A(K)
  CALL SHIFTSUM $ IF (SUM.EQ.0) 30, 25
25 IDATA(L1) = IDATA(L1) + 1
  IDATA(J+K) = SUM
  IDATA(JO + IDATA(L1)) = K
30 CONTINUE
  RETURN
  END
C

```

```

ROW000001
ROW000002
ROW000003
ROW000004
ROW000005
ROW000006
ROW000007
ROW000008
ROW000009
ROW000010
ROW000011
ROW000012
ROW000013
ROW000014
ROW000015
ROW000016
ROW000017
ROW000018
ROW000019
ROW000020
ROW000021
ROW000022
ROW000023
ROW000024
ROW000025
ROW000026
ROW000027
ROW000028
ROW000029
ROW000030
ROW000031

```


COL00001
COL00002
COL00003
COL00004
COL00005
COL00006
COL00007
COL00008
COL00009
COL00010
COL00011
COL00012
COL00013
COL00014
COL00015
COL00016
COL00017
COL00018
COL00019
COL00020
COL00021
COL00022
COL00023
COL00024
COL00025
COL00026
COL00027

```

SUBROUTINE COLSUM (K)
C
C   COMPUTE THE SUM OF EACH COLUMN OF THE MATRIX APPROPRIATELY MASKED
C
COMMON/BLOCKA/A(144)/BLOCKB/IDATA(1761)/BLOCKC/M,N,DUMMY(146)
TYPE LOGIC5 (3) A, BIT, MASK
DIMENSION MASKI(3)
EQUIVALENCE (MASKI,MASK)
GO TO (5,10,15) K
5 DO 6 I=1,3
6 MASKI(I)=IDATA(582+I) $ L=291 $ J=435 $ KO=3 $ GO TO 20
10 DO 11 I=1,3
11 MASKI(I)=IDATA(1169+I) $ L=878 $ J=1022 $ KO=590 $ GO TO 20
15 DO 16 I = 1,3
16 MASKI(I)=IDATA(1756+I) $ L=1465 $ J=1609 $ KO=1177
20 IDATA(KO) = 0
DO 45 I = 1, N $ BIT = BIT/I $ IDATA(L+I) = 0
IF(BIT + MASK.EQ.MASK) 45,25
25 DO 35 IO = 1,M
IF (IDATA(KO+IO).EQ.0.OR.BIT*A(IO).EQ.0) 35,30
30 IDATA(L+I) = IDATA(L+I) + 1
35 CONTINUE
IF(IDATA(L+I).EQ.0) 45,40
40 IDATA(KO) = IDATA(KO) + 1 $ IDATA(J+IDATA(KO)) = 1
45 CONTINUE
RETURN
END

```



```

C      SUBROUTINE BOUND (I)
C
C      COMPUTE A LOWER BOUND ON THE ALPHA-WIDTH OF A CLASS OF MATRICES
C      IF ALPHA EQUALS ONE, A GREATEST LOWER BOUND IS COMPUTED
C
COMMON/BLOCKB/IDATA(1761)/BLOCKC/I1,I2,ALPHA,EPSILON,RT(144)
TYPE INTEGER ALPHA, EPSILON, E, F, QO, QF, RT
GO TO (5,10,15) I
    5 M = IDATA(2) $ N = IDATA(3) $ K = 3 $ GO TO 20
    10 M = IDATA(589) $ N = IDATA(590) $ K = 590 $ GO TO 20
    15 M = IDATA(1176) $ N = IDATA(1177) $ K = 1177
C
    20 KO = K + 144 $ KS = KO + 144 $ K1 = KS + 144
C
    IF (IDATA(KS+IDATA(K1+1)).EQ.0) 25,30
    25 PRINT 1000, I $ STOP 100
C
    30 IF (ALPHA.GT.1) 85,31
    31 EPSILON = ALPHA*M/IDATA(KS + IDATA(K1+1)) $ F = EPSILON $ E =
      QO = (-ALPHA)*M
      DO 35 J = 1,M
    35 QO = QO + IDATA(K+IDATA(KO+J))
    40 QF = QO
    45 F = F + 1
      IF (F.GT.N) 65,50
    50 QF = QF + E - IDATA(KS+IDATA(K1+F))
    55 IF (QF.LT.0) 60,45
    60 EPSILON = EPSILON + 1 $ F = EPSILON $ E = 0
      IF (EPSILON.GT.N) 80,61
    61 QO = QO + IDATA(KS + IDATA(K1 + EPSILON)) $ GO TO 40
    65 QF = QO $ F = EPSILON $ E = E + 1 $ IF (E.GT.M) 75,66
    66 QF = QF + ALPHA - IDATA(K + IDATA(KO + E)) $ IF (QF.LT.0) 60,70
    70 IF (E.GT.M) 75,55
    75 RETURN
    80 EPSILON = I2*2 $ RETURN
C

```

BND000001
 BND000002
 BND000003
 BND000004
 BND000005
 BND000006
 BND000007
 BND000008
 BND000009
 BND000010
 BND000011
 BND000012
 BND000013
 BND000014
 BND000015
 BND000016
 BND000017
 BND000018
 BND000019
 BND000020
 BND000021
 BND000022
 BND000023
 BND000024
 BND000025
 BND000026
 BND000027
 BND000028
 BND000029
 BND000030
 BND000031
 BND000032
 BND000033
 BND000034
 BND000035
 BND000036

BND000037
BND000038
BND000039
BND000040
BND000041
BND000042
BND000043
BND000044
BND000045
BND000046

```
85 IBND = 0 $ DO 90 J = 1,M
90 IBND = IBND + RT(IDATA(KO+J)) $ ITEST = 0
    DO 95 J = 1,N
    ITEST = ITEST + IDATA(KS+IDATA(KI+J))
    IF (ITEST.GE.IRND) 100,95
95 CONTINUE $ EPSILON = 2*I2 $ RETURN
100 EPSILON = J $ RETURN
C 1000 FORMAT(48H1ALGORITHM HAS PROCEEDED PAST TERMINATION POINT.10X,I1)
    END
```


BLOCKF	IDENT	RANDGEN
	3	
BLOCKG	COMMON	WORD(3)
	1	
	R	
RANDGEN	ENTRY	RANDGEN
	SLJ	**
	SIU	ENDING
	1	
	END	2
	ENG	0
A1	LDA	R
	LLS	10
	ADD	R
	INA	101
	STA	R
	1	WORD
	1	A1
ENDING	1	**
	SLJ	RANDGEN
	END	

THREE RANDOM NUMBERS ARE
GENERATED BY THE FORMULA
 $X(I+1) = X(I) * 2 * 10 + X(I) + 101$

GEN00001
GEN00002
GEN00003
GEN00004
GEN00005
GEN00006
GEN00007
GEN00008
GEN00009
GEN00010
GEN00011
GEN00012
GEN00013
GEN00014
GEN00015
GEN00016
GEN00017
GEN00018
GEN00019
GEN00020

TIM000001
TIM000002
TIM000003
TIM000004
TIM000005
TIM000006
TIM000007
TIM000008
TIM000009
TIM000010
TIM000011
TIM000012
TIM000013
TIM000014
TIM000015
TIM000016

READ THE CURRENT VALUE OF THE
REAL TIME CLOCK

IDENT	TIME	TIME
ENTRY	**	TIME
SLJ	*	
LDA	24	
ALS	A1	
SAU	1	
INA	TIME	
SAU	24	
LDA	24	
ALS	A2	
SAU	0	
ENQ	=00	
LDA	**	
STA	TIME	
SLJ		
END		

7

TIME

A1

A2

IDENT	SHIFTSUM
ENTRY	SHIFTSUM
BLOCK	4
COMMON	ROW(3),SUM
SHIFTSUM	**
SIU	ENDING
SIL	ENDING
ENA	0
STA	SUM
FNI	2
LDQ	ROW
ENI	47
QJP	*+1
SLJ	*+2
RAO	SUM
IJP	*+1
SLJ	*+2
QLS	1
SLJ	A2
IJP	A1
ENI	**
FNI	**
SLJ	SHIFTSUM
END	

SUM000001
SUM000002
SUM000003
SUM000004
SUM000005
SUM000006
SUM000007
SUM000008
SUM000009
SUM000010
SUM000011
SUM000012
SUM000013
SUM000014
SUM000015
SUM000016
SUM000017
SUM000018
SUM000019
SUM000020
SUM000021
SUM000022
SUM000023
SUM000024

THIS ROUTINE COMPUTES
THE NUMBER OF 1-BITS IN
UP TO 3 CONSECUTIVE LOCATIONS
OF MEMORY.

MTH000001
MTH000002
MTH000003
MTH000004
MTH000005
MTH000006
MTH000007
MTH000008
MTH000009
MTH000010
MTH000011
MTH000012
MTH000013
MTH000014
MTH000015

THIS ROUTINE PERFORMS ALL TYPE
LOGIC5 MATH OPERATIONS.

LOAD EACH WORD OF THE
ACCUMULATOR WITH THE SAME
INTEGER CONSTANT.

ACC	Q1Q000500	IDENT	Q1QMATH
		REM	0,0,0
		OCT	Q1Q000500
		ENTRY	**
		SLJ	*
		LDA	INITIAL
		RTJ	
		NOP	
		SAU	*+1
+		LDA	**
+		STA	ACC+2
		ENA	0
		STA	ACC
		STA	ACC+1
		SLJ	Q1Q000500

MTH000016
MTH000017
MTH000018
MTH000019
MTH000020
MTH000021
MTH000022
MTH000023
MTH000024
MTH000025
MTH000026
MTH000027
MTH000028
MTH000029
MTH000030
MTH000031
MTH000032

LOAD THE ACCUMULATOR WITH
A LOGIC5 CONSTANT.

ACC	Q1Q000550	ENTRY	Q1Q000550
		SLJ	**
		LDA	*
		RTJ	INITIAL
		NOP	
		SAU	A1
		INA	1
		SAU	A2
		INA	1
		SAU	A3
		LDA	**
		STA	ACC
		LDA	**
		STA	ACC+1
		LDA	**
		STA	ACC+2
		SLJ	Q1Q000550

MTH000033
 MTH000034
 MTH000035
 MTH000036
 MTH000037
 MTH000038
 MTH000039
 MTH000040
 MTH000041
 MTH000042
 MTH000043
 MTH000044
 MTH000045
 MTH000046
 MTH000047
 MTH000048
 MTH000049

LOAD THE ACCUMULATOR WITH
 THE COMPLEMENT OF A
 LOGIC5 CONSTANT.
 (UNARY -OPERATOR)

Q1Q01550	ENTRY	Q1Q01550
+	SLJ	**
	LDA	*
	RTJ	INITIAL
	NOP	
	SAU	B1
	INA	1
	SAU	B2
	INA	1
	SAU	B3
B1	LAC	**
	STA	ACC
B2	LAC	**
	STA	ACC+1
B3	LAC	**
	STA	ACC+2
	SLJ	Q1Q01550

MTH000050
 MTH000051
 MTH000052
 MTH000053
 MTH000054
 MTH000055
 MTH000056
 MTH000057
 MTH000058
 MTH000059
 MTH000060
 MTH000061
 MTH000062
 MTH000063
 MTH000064
 MTH000065
 MTH000066
 MTH000067
 MTH000068
 MTH000069

FORM THE LOGICAL SUM OF
 THE ACCUMULATOR AND A
 LOGIC5 CONSTANT.
 (+OPERATOR)

Q1Q02550	ENTRY	Q1Q02550	SLJ	Q1Q02550	**
	LDA			*	INITIAL
	RTJ				C1
	NOP				1
	SAL				C2
	INA				1
	SAU				C3
	INA				ACC
	SAL				**
	LDA				ACC
	SST				ACC+1
	STA				**
	LDA				ACC+1
	SST				ACC+2
	STA				**
	LDA				ACC+2
	SST				Q1Q02550
	STA				
	SLJ				

MTH000098
MTH000099
MTH000100
MTH000101
MTH000102
MTH000103
MTH000104
MTH000105
MTH000106
MTH000107
MTH000108
MTH000109
MTH000110
MTH000111
MTH000112
MTH000113
MTH000114
MTH000115
MTH000116
MTH000117
MTH000118
MTH000119
MTH000120
MTH000121
MTH000122
MTH000123
MTH000124
MTH000125
MTH000126
MTH000127

CLEAR BITS OF THE ACCUMULATOR
WHERE THERE ARE CORRESPONDING
1-BITS IN A LOGIC5
WORD. LOAD THE A-REGISTER
WITH A NON-ZERO PART OF
THE RESULT IF IT EXISTS.
OTHERWISE LOAD THE
A-REGISTER WITH ZERO.
(--OPERATOR)

ENTRY	Q1Q03550	Q1Q03550	
LDA	**	**	
RTJ	*	INITIAL	
NOP			
SAL	E1		
INA	1		
SAU	E2		
INA	1		
SAL	E3		
SIU	E4		
ENI	0		
LDA	ACC		
SCL	**		
AJP	*+1		
ENI	0		
STA	ACC		
LDA	ACC+1		
SCL	**		
STA	ACC+1		
AJP	*+1		
ENI	1		
LDA	ACC+2		
SCL	**		
AJP	*+1		
ENI	2		
STA	ACC+2		
LDA	ACC		
ENI	**		
SLJ	Q1Q03550		

MTH00128
 MTH00129
 MTH00130
 MTH00131
 MTH00132
 MTH00133
 MTH00134
 MTH00135
 MTH00136
 MTH00137
 MTH00138
 MTH00139
 MTH00140
 MTH00141
 MTH00142
 MTH00143
 MTH00144
 MTH00145
 MTH00146
 MTH00147

FORM THE LOGICAL PRODUCT
 OF THE ACCUMULATOR AND
 A LOGIC5 WORD.
 (*OPERATOR)

Q1Q04550	ENTRY	Q1Q04550
	SLJ	**
	LDA	*
	RTJ	INITIAL
	NOP	
	SAU	F1
	INA	1
	SAL	F2
	INA	1
	SAU	F3
	LDQ	ACC
	LDL	**
F1	STA	ACC
F2	LDQ	ACC+1
	LDL	**
	STA	ACC+1
	LDQ	ACC+2
F3	LDL	**
	STA	ACC+2
	SLJ	Q1Q04550

MTH00148
MTH00149
MTH00150
MTH00151
MTH00152
MTH00153
MTH00154
MTH00155
MTH00156
MTH00157
MTH00158
MTH00159
MTH00160
MTH00161
MTH00162
MTH00163
MTH00164
MTH00165
MTH00166
MTH00167
MTH00168
MTH00169
MTH00170
MTH00171
MTH00172
MTH00173
MTH00174

GENERATE A 1-BIT IN
THE BIT POSITION OF
THE ACCUMULATOR CORRES-
PONDING TO THE INTEGER
ARGUMENT COUNTING FROM
THE MOST SIGNIFICANT
BIT POSITION
(/OPERATOR)

ENTRY	Q1Q05500	Q1Q05500	
SLJ	**	**	
SIU	1	G3	
ENA		0	
STA		ACC	
STA		ACC+1	
STA		ACC+2	
LDA		Q1Q05500	
NOP		INITIAL	
RTJ			
NOP		*+1	
SAU		**	
LDA		-1	
ENI	1	1	
INI	1	-48	
INA	2	G1	
AJP		48	
INA		=SG2	
STA		48	
ENA		G2	
SUB		*+1	
SAL		1	
ENA		**	
ALS		ACC	
STA	1	**	
ENI	1	Q1Q05500	
SLJ			

MTH00175
 MTH00176
 MTH00177
 MTH00178
 MTH00179
 MTH00180
 MTH00181
 MTH00182
 MTH00183
 MTH00184
 MTH00185
 MTH00186
 MTH00187
 MTH00188
 MTH00189

STORE ZERO IN EACH
 PART OF A LOGIC5 WORD.

Q1Q10050	ENTRY	Q1Q10050
+	SLJ	**
	LDA	*
	RTJ	INITIAL
	NOP	
	SAU	H1
	INA	1
	SAL	H1
	INA	1
	SAU	H2
	ENA	0
H1	STA	**
	STA	**
H2	STA	**
	SLJ	Q1Q10050

MTH00190
MTH00191
MTH00192
MTH00193
MTH00194
MTH00195
MTH00196
MTH00197
MTH00198
MTH00199
MTH00200
MTH00201
MTH00202
MTH00203
MTH00204
MTH00205
MTH00206

STORE THE ACCUMULATOR IN
A LOGIC5 WORD.

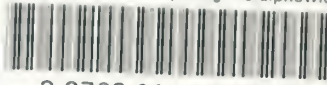
Q1Q10550	ENTRY	Q1Q10550
	SLJ	**
	LDA	* INITIAL
	RTJ	
	NOP	
+	SAL	J1
	INA	1
	SAL	J2
	INA	1
	SAL	J3
J1	LDA	ACC
	STA	**
J2	LDA	ACC+1
	STA	**
J3	LDA	ACC+2
	STA	**
	SLJ	Q1Q10550

MTH00207
MTH00208
MTH00209
MTH00210
MTH00211
MTH00212
MTH00213

INITIAL	SLJ	**
	ALS	24
	INA	-1
	SAU	*+1
+	ENA	7 **
	SLJ	INITIAL
	END	

mess674

An algorithm for computing the alphawidt



3 2768 001 00978 0

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